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A STABILIZED FINITE ELEMENT METHOD FOR GENERALIZED INCOMPRESSIBLE FLOW PROBLEMS*

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Abstract. *A new stabilized finite element method is introduced for an incompressible flow problem modelled by the Oseen equation (or linearized Navier-Stokes equation) containing a dominating zeroth order term. The method consists in subtracting a mesh dependent term from the formulation without compromising consistency, which also allows the use of equal order interpolation for both velocity and pressure. The design of this mesh dependent term, as well as the stabilization parameter involved, are suggested by bubble condensation. Numerical stability and optimal order error estimates are proven in the natural norms for velocity and pressure. Moreover, an $L^2(\Omega)$ error estimate for the velocity is proved, and in this estimate the difference between dominating diffusion and dominating convection is explicited. Numerical experiments confirming these theoretical results are presented.*

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1 INTRODUCTION

The Navier-Stokes equation constitutes a major challenge in applied mathematics. Specifically, its numerical solution presents two major difficulties, namely, the need for a compatibility condition (the inf-sup condition, see[1,2]) relating the discrete spaces used to approximate the velocity field $\mathbf{u}$ and the pressure p, and the treatment of the spurious modes generated by the convective term. For both these aspects, several solutions have been proposed in the last two decades. The convective terms have been treated by appropriate upwinding strategies (cf.[1,3,4] and the references therein), or stabilized finite element methods (cf.[5,6] among others). On the other hand, the inf-sup condition may be treated directly (cf.[1,2] and the references therein), or circumvented via stabilized finite element methods (cf.[7–11] in the context of a Stokes flow).

On the other hand, if we are dealing with the time discretization of the Navier-Stokes equation, and we choose the "classical" approach (i.e., discretizing in time by time-advancing finite differences) we have different choices for the scheme (for a resume of these techniques, see[3]). A common fact of all these techniques is the presence of a zeroth order term of type $\frac{1}{\Delta t}\mathbf{u}$, where Δt is the time step (usually very small), and $\mathbf{u}$ is the unknown velocity field. In the late nineties, several works concerning stabilization procedures for problems with zeroth order terms (or reaction terms), were proposed (see, e.g.[12,13] and the recent paper[14] where edge stabilization has been proposed for a scalar convection-diffusion-reaction problem). In particular, in[15–17] the connection between stabilized finite element methods and Galerkin methods enriched with bubble functions was used to derive a new family of stabilized finite element method, namely, the Unusual Stabilized Finite Element Method (USFEM), which are particularly suited for treating problems with dominating reaction.

In this work we continue the work from[18] where the method was originally proposed for a problem including convection, and give some new error estimates and numerical experiments. For completeness, we review the analysis of the method in Section 2, where an error estimate is derived for the standard norms of velocity and pressure. Moreover, a new approximation result is presented at the end of this section, now for the $L^2(\Omega)$ norm of the velocity error, obtained by modifying a duality argument. The estimate is suboptimal in the convection dominated case, which has been observed for SDFEM and GLS methods (see,[19–21] and specially the introduction in[22]). Finally, in Section 3 we report some numerical experiments that confirm our approximation results, and show some extra features of the method.

2 THE FINITE ELEMENT METHOD

First, we present the problem of interest. Let Ω be a bounded open subset of $\mathbb{R}^2$, $\mathbf{f} \in [L^2(\Omega)]^2$, σ a positive real number (typically, $\sigma \approx \frac{1}{\Delta t}$ where Δt is the time step in a time discretization procedure), and $\mathbf{a} : \Omega \to \mathbb{R}^2$ a vectorial function such that $\nabla \cdot \mathbf{a} = 0$ in Ω (this function $\mathbf{a}$ may be interpreted as the velocity field in the previous time step). Our generalized incompressible flow problem reads: *Find* $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ *such that*

$$\begin{aligned} \sigma\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{a}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega\,, \\ \nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega\,, \\ \mathbf{u} = 0 \quad & \text{on } \partial\Omega\,, \end{aligned} \tag{1}$$

where $L_0^2(\Omega) \stackrel{\text{def}}{=} \{q \in L^2(\Omega) : (q,1)_\Omega = 0\}$, and $(\cdot,\cdot)_D$ denotes the L^2 inner product in $L^2(D)$ (or in $L^2(D)^2, L^2(D)^{2\times 2}$, when necessary). Also, by $\|\cdot\|_{l,D}$ and $|\cdot|_{l,D}$ we will denote the $H^l(D)$ norm and seminorm, respectively, with the usual convention $H^0(D)=L^2(D)$.

From now on, let us suppose that Ω is a polygonal domain in $\mathbb{R}^2$, and let $\mathcal{T}_h$ be a triangulation of Ω constituted by triangles (or quadrilaterals) which are shape regular. Let h_K be the usual element diameter, and denote $h \stackrel{\text{def}}{=} \max\{h_K : K \in \mathcal{T}_h\}$. We suppose from now on that $h \le 1$. Now, for $k \ge 1$, let V_k be the space of piecewise polynomial functions given by

$$V_k \stackrel{\text{def}}{=} \left\{ v \in \mathcal{C}^0(\overline{\Omega})/v\big|_K \in R^k(K)\,, \forall K \in \mathcal{T}_h \right\}\,.$$

Here, $R^k(K) = P^k(K)$ for triangular elements and $R^k(K) = \{p \circ F_K^{-1} \,/\, p \in Q^k(\hat{K})\}$ for quadrilateral elements, where F_K stands for the usual transformation mapping the reference element $\hat{K}$ onto K.

In weak form, this problem reads: Find $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ such that

$$\mathbf{A}((\mathbf{u},p),(\mathbf{v},q)) = (\mathbf{f},\mathbf{v})_\Omega \qquad \forall (\mathbf{v},q) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega),$$

where

$$\mathbf{A}((\mathbf{u},p),(\mathbf{v},q)) \stackrel{\text{def}}{=} \sigma(\mathbf{u},\mathbf{v})_\Omega + \nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + (\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v})_\Omega - (p,\nabla\cdot\mathbf{v})_\Omega + (q,\nabla\cdot\mathbf{u})_\Omega. \tag{2}$$

Our stabilized finite element method reads: *Find* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *such that:*

$$\mathbf{B}((\mathbf{u}_h,p_h),(\mathbf{v},q)) = \mathbf{F}(\mathbf{v},q) \qquad \forall\,(\mathbf{v},q) \in \mathbf{V}_h \times Q_h\,, \tag{3}$$

where $\mathbf{V}_h \stackrel{\text{def}}{=} [V_k \cap H_0^1(\Omega)]^2$ and $Q_h \stackrel{\text{def}}{=} V_l \cap L_0^2(\Omega)$, $k, l \ge 1$, $\mathbf{B}$ and $\mathbf{F}$ are given by

$$\begin{aligned} \mathbf{B}((\mathbf{u}_h,p_h),(\mathbf{v},q)) &\stackrel{\text{def}}{=} \mathbf{A}((\mathbf{u}_h,p_h),(\mathbf{v},q)) + \sum_{K\in\mathcal{T}_h} (\delta_K \nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v})_K \\ &- \sum_{K\in\mathcal{T}_h} (\sigma\mathbf{u}_h - \nu\Delta\mathbf{u}_h + \mathbf{a}\cdot\nabla\mathbf{u}_h + \nabla p_h, \tau_K(\sigma\mathbf{v} - \nu\Delta\mathbf{v} - \mathbf{a}\cdot\nabla\mathbf{v} - \nabla q))_K\,, \end{aligned} \tag{4}$$

$$\mathbf{F}(\mathbf{v},q) \stackrel{\text{def}}{=} (\mathbf{f},\mathbf{v})_\Omega - \sum_{K\in\mathcal{T}_h} (\mathbf{f}, \tau_K(\sigma\mathbf{v} - \nu\Delta\mathbf{v} - \mathbf{a}\cdot\nabla\mathbf{v} - \nabla q))_K. \tag{5}$$

Here, the stabilization parameters τ_K and δ_K are given by

$$\tau_K \stackrel{\text{def}}{=} \frac{h_K^2}{\sigma h_K^2 \xi(Pe_K^1) + \frac{4\nu}{m_k}\xi(Pe_K^2)}\,, \tag{6}$$

$$\delta_K \stackrel{\text{def}}{=} \lambda\,|\mathbf{a}(x)|_2\,h_K\,\min\{1, Pe_K^2\}\,, \tag{7}$$

where $\lambda \geq 0$ and

$$Pe_K^1 = \frac{4\nu}{m_k \sigma h_K^2}\,, \tag{8}$$

$$Pe_K^2 = \frac{m_k |\mathbf{a}|_2 h_K}{4\nu}\,, \tag{9}$$

$$|\mathbf{a}(x)|_2 \stackrel{\text{def}}{=} (|a_1|^2 + |a_2|^2)^{1/2}\,, \tag{10}$$

$$m_k = \min\{\frac{1}{3}, C_k\}\,, \tag{11}$$

$$C_k h_K^2\,\|\Delta v\|_{0,K}^2 \leq \|\nabla v\|_{0,K}^2 \qquad \forall\, v \in V_k\,, \tag{12}$$

$$\xi(\lambda) = \max\{\lambda, 1\}\,. \tag{13}$$

Remark 1 *The design of the stabilization parameter τ_K has been suggested by bubble condensation, following very closely the arguments given in.[17,23] The least-squares parameter δ_K is the one from.[6] In the case of a generalized Stokes problem ($\mathbf{a} = \mathbf{0}$) we recover the method from.[17] Now, in the case of a pure Oseen equation $\sigma = 0$, we recover the "plus" formulation from[6] with a stabilization parameter which satisfies $\tau_{FF} \leq \tau_K \leq 2\tau_{FF}$, where τ_{FF} denotes the stabilization parameter proposed in,[6] given by*

$$\tau_{FF} \stackrel{\text{def}}{=} \frac{h_K}{2|\mathbf{a}(x)|_2}\min\{1, Pe_K^2\}\,.\square$$

Remark 2 *In[24] the orthogonal subscales approach was applied to a related problem containing a Coriolis terms and a zeroth order term. The resulting formulation involves stabilization parameters with free constants to be set. The performance of the method depends on how these constants are chosen.* □

2.1 The stability of the method

Throughout all this section (and the following one), C will denote a positive constant independent of h (but who may depend on the physical coefficients), whose value may vary whenever it is written in two different places.

The following lemma provides the positive-definiteness of the stiffness matrix associated with our method it's proof is given in[18] .

Lemma 1 *There exists a constant C_Ω, depending only on Ω, such that*

$$\mathbf{B}((\mathbf{v},q),(\mathbf{v},q)) \;\geq\; C_\Omega \nu \,\|\mathbf{v}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \left\{ \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v} + \nabla q)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|_{0,K}^2 \right\}, \quad (14)$$

for all $(\mathbf{v},q) \in \mathbf{V}_h \times Q_h$.

Remark 3 *From Lemma 1 above, a first error estimate involving the mesh dependent norm appearing in the right hand side of* (14) *may be given (see[17] for an estimate of this kind in the case of a Stokes flow). However, this estimate has a couple of important drawbacks (see[17] for a discussion), and that is why in the rest of this section we follow an alternative approach.* □

Now, in order to prove our main result in stability, namely the inf-sup condition for $\mathbf{B}$, we define the following mesh-dependent norm:

$$|||(\mathbf{v},q)|||_h \;\stackrel{\text{def}}{=}\; \left\{ \|\mathbf{v}\|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \left[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v} + \nabla q)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|_{0,K}^2 \right] \;+\; \|q\|_{0,\Omega}^2 \right\}^{1/2}, \quad (15)$$

for all $(\mathbf{v},q) \in \mathbf{V}_h \times Q_h$.

We now state the main stability result.

Theorem 2 *There exists a constant $\beta = \beta(\sigma,\mathbf{a},\nu)$, independent of h, such that*

$$\sup_{\theta\neq(\mathbf{w},t)\in\mathbf{V}_h\times Q_h} \frac{\mathbf{B}((\mathbf{u},p),(\mathbf{w},t))}{|||(\mathbf{w},t)|||_h} \;\geq\; \beta\, |||(\mathbf{u},p)|||_h\,,$$

for all $(\mathbf{u},p) \in \mathbf{V}_h \times Q_h$.

Proof. - Let $(\mathbf{u},p) \in \mathbf{V}_h \times Q_h$. Since $p \in L_0^2(\Omega)$, there exists $\mathbf{v} \in [H_0^1(\Omega)]^2$ such that $\nabla\cdot\mathbf{v} = -p$ in Ω and $\|\mathbf{v}\|_{1,\Omega} \leq C\,\|p\|_{0,\Omega}$. Now, let $\mathbf{v}_h$ be the Clément interpolate of $\mathbf{v}$ (cf.[1,25]), which satisfies

$$\|\mathbf{v} - \mathbf{v}_h\|_{0,K} \leq C\,h_K\,\|\mathbf{v}\|_{1,V(K)}\,, \quad (16)$$

$$\|\mathbf{v}_h\|_{1,\Omega} \leq C\,\|\mathbf{v}\|_{1,\Omega}\,, \quad (17)$$

where $V(K)$ is the set of elements in $\mathcal{T}_h$ who share at least one node with K. After some manipulations (for the datails, see[18]), we arrive at

$$\mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0)) \geq -\,C^{**} \left\{ \|\mathbf{u}\|_{1,\Omega}^2 \,+\, \sum_{K\in\mathcal{T}_h} \left[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p)\|_{0,K}^2 + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|_{0,K}^2 \right] \right\} \\ +\; C^{*}\,\|p\|_{0,\Omega}^2\,, \quad (18)$$

where C^* and C^{**} are positive constants, independents of h.

In this form, if we set $(\mathbf{z},q) \stackrel{\text{def}}{=} (\mathbf{u}+\gamma\mathbf{v}_h,p)\,,\ \gamma>0$, we have by the bilinearity of $\mathbf{B}$ and Lemma 1

$$\begin{aligned}
\mathbf{B}((\mathbf{u},p),(\mathbf{z},q)) &= \mathbf{B}((\mathbf{u},p),(\mathbf{u},p)) \,+\, \gamma\,\mathbf{B}((\mathbf{u},p),(\mathbf{v}_h,0))\\
&\geq C_\Omega\nu\,\|\mathbf{u}\|^2_{1,\Omega} + \sum_{K\in\mathcal{T}_h}\Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|^2_{0,K} + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|^2_{0,K}\Big]\\
&\quad -\gamma C^{**}\left\{\|\mathbf{u}\|^2_{1,\Omega} + \sum_{K\in\mathcal{T}_h}\Big[\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|^2_{0,K} + \|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|^2_{0,K}\Big]\right\}\\
&\quad +\gamma\,C^*\,\|p\|^2_{0,\Omega}\\
&\geq C\,|||(\mathbf{u},p)|||^2_h\,,
\end{aligned}\tag{19}$$

where $C>0$ is independent of h if γ is chosen small enough. Finally, using that $h\leq 1$ and $\|\mathbf{v}\|_{1,\Omega}\leq C\,\|p\|_{0,\Omega}$, we obtain

$$\begin{aligned}
|||(\mathbf{z},q)|||^2_h &\leq 2\,\|\mathbf{u}\|^2_{1,\Omega} + 2\sum_{K\in\mathcal{T}_h}\|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{u}+\nabla p)\|^2_{0,K} + 2\sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\mathbf{u}\|^2_{0,K} + \|p\|^2_{0,\Omega}\\
&\quad + 2\gamma^2\left[\|\mathbf{v}_h\|^2_{1,\Omega} + \sum_{K\in\mathcal{T}_h}\overline{\tau_K}^{1/2}\|\mathbf{a}\cdot\nabla\mathbf{v}_h\|^2_{0,K} + \sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\mathbf{v}_h\|^2_{0,K}\right]\\
&\leq C\,|||(\mathbf{u},p)|||^2_h\,,
\end{aligned}$$

which, together with (19), finish the proof. □

2.2 Error Analysis

Let k,l be integers with $k,l\geq 1$. We use the Lagrange interpolation operator $\mathbf{I}_h^k:(C^0(\overline{\Omega}))^2\to[V_k]^2$, we denote $\tilde{\mathbf{u}}_h \stackrel{\text{def}}{=} \mathbf{I}_h^k(\mathbf{u})$, define the interpolation error $\eta^{\mathbf{u}} \stackrel{\text{def}}{=} \mathbf{u}-\tilde{\mathbf{u}}_h$, and we have (cf.[26])

$$|\eta^{\mathbf{u}}|_{m,K}\;\leq\;Ch_K^{s-m}\,|\mathbf{u}|_{s,K}\,,\tag{20}$$

if $\mathbf{u}\in H^s(K)^2$ for all $K\in\mathcal{T}_h$, with $0\leq m\leq 2$ and $\max\{m,2\}\leq s\leq k+1$. Now, for the pressure we define $\tilde{p}_h$ as being the Clément interpolate of p. Denoting now by $\eta^p \stackrel{\text{def}}{=} p-\overline{p_h}$, where

$$\overline{p_h} \stackrel{\text{def}}{=} \tilde{p}_h - \frac{1}{|\Omega|}\,(\tilde{p}_h,1)_\Omega\;\in\;L^2_0(\Omega)\,,$$

we have (cf.[1])

$$\|\eta^p\|_{0,\Omega}\leq Ch^t\,\|p\|_{t,\Omega}\,,\tag{21}$$

$$|\eta^p|_{1,K}\leq Ch^{t-1}\,\|p\|_{t,V(K)}\,,\tag{22}$$

if $p\in H^t(\Omega)$, with $1\leq t\leq l+1$.

The main result concerning approximation is now stated.

Theorem 3 *Let us suppose that the solution* $(\mathbf{u},p)$ *of (1) belongs to* $(H^{k+1}(\Omega)\cap H_0^1(\Omega))^2 \times (H^l(\Omega)\cap L_0^2(\Omega))$*. Then, there exists* $C>0$*, independent of* h*, such that the error* $(\mathbf{e^u},e^p)\stackrel{\text{def}}{=}(\mathbf{u}-\mathbf{u}_h,p-p_h)$ *satisfies*

$$\|\mathbf{e^u}\|_{1,\Omega}\,+\,\|e^p\|_{0,\Omega}\,\le\,C\left[h^k\,|\mathbf{u}|_{k+1,\Omega}\,+\,h^l\,\|p\|_{l,\Omega}\right].$$

Proof. - Let $\mathbf{e}_h^{\mathbf{u}}\stackrel{\text{def}}{=}\mathbf{u}_h-\tilde{\mathbf{u}}_h$ and $e_h^p\stackrel{\text{def}}{=}p_h-\overline{p_h}$. From the proof of previous theorem, we see that the suppremum is attained, and then there exists $(\mathbf{v},q)\in\mathbf{V}_h\times Q_h$ such that

$$|\!|\!|(\mathbf{v},q)|\!|\!|_h\,\le\,C\,,$$

and (using that $2(\|\mathbf{e}_h^{\mathbf{u}}\|_{1,\Omega}+\|e_h\|_{0,\Omega})\le|\!|\!|(\mathbf{e}_h^{\mathbf{u}},e_h)|\!|\!|_h$)

$$2\beta\left[\|\mathbf{e}_h^{\mathbf{u}}\|_{1,\Omega}\,+\,\|e_h^p\|_{0,\Omega}\right]\,\le\,\mathbf{B}((\mathbf{e}_h^{\mathbf{u}},e_h^p),(\mathbf{v},q))\,=\,\mathbf{B}((\eta^{\mathbf{u}},\eta^p),(\mathbf{v},q))\,, \tag{23}$$

thanks to the consistency of the method. Now, for the right hand side of (23) we have by using Schwartz's inequality

$$\begin{aligned}
\mathbf{B}((\eta^{\mathbf{u}},\eta^p),(\mathbf{v},q)) &= \sigma(\eta^{\mathbf{u}},\mathbf{v})_\Omega\,+\,\nu(\nabla\eta^{\mathbf{u}},\nabla\mathbf{v})_\Omega\,+\,(\mathbf{a}\cdot\nabla\eta^{\mathbf{u}},\mathbf{v})_\Omega\\
&\quad-\,(\eta^p,\nabla\cdot\mathbf{v})_\Omega\,+\,(q,\nabla\cdot\eta^{\mathbf{u}})_\Omega\,+\sum_{K\in\mathcal{T}_h}(\nabla\cdot\eta^{\mathbf{u}},\delta_K\nabla\cdot\mathbf{v})_K\\
&\quad-\sum_{K\in\mathcal{T}_h}(\sigma\eta^{\mathbf{u}}-\nu\Delta\eta^{\mathbf{u}}+\mathbf{a}\cdot\nabla\eta^{\mathbf{u}}+\nabla\eta^p,\tau_K(\sigma\mathbf{v}-\nu\Delta\mathbf{v}-\mathbf{a}\cdot\nabla\mathbf{v}-\nabla q))_K\\
&\le\Big\{\sum_{K\in\mathcal{T}_h}\sigma\,\|\eta^{\mathbf{u}}\|_{0,K}^2\,+\,\nu\,|\eta^{\mathbf{u}}|_{1,K}^2\,+\,\|\mathbf{a}\cdot\nabla\eta^{\mathbf{u}}\|_{0,K}^2\,+\,\|\eta^p\|_{0,K}^2\,+\,\|\nabla\cdot\eta^{\mathbf{u}}\|_{0,K}^2\\
&\qquad+\|\delta_K^{1/2}\nabla\cdot\eta^{\mathbf{u}}\|_{0,K}^2\,+\,\|\tau_K^{1/2}(\sigma\eta^{\mathbf{u}}-\nu\Delta\eta^{\mathbf{u}}+\mathbf{a}\cdot\nabla\eta^{\mathbf{u}}+\nabla\eta^p)\|_{0,K}^2\Big\}^{\frac12}\\
&\quad\cdot\Big\{\sum_{K\in\mathcal{T}_h}\sigma\,\|\mathbf{v}\|_{0,K}^2\,+\,\nu\,|\mathbf{v}|_{1,K}^2\,+\,\|\mathbf{v}\|_{0,K}^2\,+\,\|\nabla\cdot\mathbf{v}\|_{0,K}^2\,+\,\|q\|_{0,K}^2\\
&\qquad+\|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|_{0,K}^2\,+\,\|\tau_K^{1/2}(\sigma\mathbf{v}-\nu\Delta\mathbf{v}-\mathbf{a}\cdot\nabla\mathbf{v}-\nabla q)\|_{0,K}^2\Big\}^{\frac12}.
\end{aligned}$$

Now, applying inequality (12) for the $-\nu^2\tau_K\|\Delta\mathbf{v}\|_{0,K}^2$ term and $\sigma\overline{\tau_K}\le1$, previous inequality becomes

$$\mathbf{B}((\eta^{\mathbf{u}},\eta^p),(\mathbf{v},q)) \le C\,\Big\{\max\{\sigma,\nu,\|\mathbf{a}\|^2_{\infty,\Omega},\|\mathbf{a}\|_{\infty,\Omega}h,\frac{1}{\nu}\}\sum_{K\in\mathcal{T}_h}\Big[\|\eta^{\mathbf{u}}\|^2_{0,K} + |\eta^{\mathbf{u}}|^2_{1,K} + \|\eta^p\|^2_{0,K} + h_K^2\|\Delta\eta^{\mathbf{u}}\|^2_{0,K} + h_K^2|\eta^p|^2_{1,K}\Big]\Big\}^{\frac{1}{2}} \cdot\Big\{\max\{\sigma,\nu,1\}\sum_{K\in\mathcal{T}_h}\Big[\|\mathbf{v}\|^2_{0,K} + |\mathbf{v}|^2_{1,K} + \|q\|^2_{0,K} + \|\delta_K^{1/2}\nabla\cdot\mathbf{v}\|^2_{0,K} + \|\mathbf{v}\|^2_{0,K} + |\mathbf{v}|^2_{1,K} + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\mathbf{v}+\nabla q)\|^2_{0,K}\Big]\Big\}^{\frac{1}{2}}\,.$$

Now, the second term in the product above is smaller than $C\,|||(\mathbf{v},q)|||_h$ and hence it is bounded by a constant. Therefore, applying interpolation inequalities (20), (21) and (22) we arrive at

$$\begin{aligned}\mathbf{B}((\eta^{\mathbf{u}},\eta^p),(\mathbf{v},q)) &\le C(\sigma,\nu,\mathbf{a})\Big\{\sum_{K\in\mathcal{T}_h}\Big[\|\eta^{\mathbf{u}}\|^2_{0,K} + |\eta^{\mathbf{u}}|^2_{1,K} + h_K^2\|\Delta\eta^{\mathbf{u}}\|^2_{0,K}\Big] + \|\eta^p\|^2_{0,\Omega} + h^2|\eta^p|^2_{1,\Omega}\Big\}^{\frac{1}{2}}\\ &\le C\,(h^k\,|\mathbf{u}|_{k+1,\Omega} + h^l\,\|p\|_{l,\Omega})\,. \end{aligned} \tag{24}$$

Finally, applying triangle inequality and (24) we arrive at

$$\begin{aligned}&\|\mathbf{e}^{\mathbf{u}}\|_{1,\Omega} + \|e^p\|_{0,\Omega} \le |||(\mathbf{e}_h^{\mathbf{u}},e_h^p)|||_h + |||(\eta^{\mathbf{u}},\eta^p)|||_h\\ &\le \frac{C}{\beta}\,(h^k\,|\mathbf{u}|_{k+1,\Omega} + h^l\,\|p\|_{l,\Omega})\\ &\quad+\Big\{\|\eta^{\mathbf{u}}\|^2_{1,\Omega} + \|\eta^p\|^2_{0,\Omega} + \sum_{K\in\mathcal{T}_h}\|\delta_K^{1/2}\nabla\cdot\eta^{\mathbf{u}}\|^2_{0,K} + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla\eta^{\mathbf{u}}+\nabla\eta^p)\|^2_{0,K}\Big\}^{\frac{1}{2}}\\ &\le \frac{C}{\beta}\,(h^k\,|\mathbf{u}|_{k+1,\Omega} + h^l\,\|p\|_{l,\Omega})\\ &\quad+C\,\Big\{\|\eta^{\mathbf{u}}\|^2_{1,\Omega} + \|\eta^p\|^2_{0,\Omega} + \|\mathbf{a}\|_{\infty,\Omega}h\,|\eta^{\mathbf{u}}|^2_{1,\Omega} + \|\mathbf{a}\|^2_{\infty,\Omega}\frac{h^2}{\nu}|\eta^{\mathbf{u}}|^2_{1,\Omega} + \frac{h^2}{\nu}\,|\eta^p|^2_{1,\Omega}\Big\}^{\frac{1}{2}}\\ &\le \frac{C}{\beta}\,(h^k\,|\mathbf{u}|_{k+1,\Omega} + h^l\,\|p\|_{l,\Omega}) + C\,\big\{\|\eta^{\mathbf{u}}\|^2_{1,\Omega} + \|\eta^p\|^2_{0,\Omega} + h^2\,|\eta^p|^2_{1,\Omega}\big\}^{\frac{1}{2}}\,,\end{aligned}$$

and the proof follows by applying interpolation inequalities (20), (21) and (22) once again.□

2.3 An error estimate in $L^2(\Omega)$-norm

In the sequel, we will assume that the solution of the following auxiliary dual problem :

$$\begin{aligned} \sigma\boldsymbol{\varphi} - \nu\Delta\boldsymbol{\varphi} - \mathbf{a}\cdot\nabla\boldsymbol{\varphi} - \nabla\pi = \mathbf{u} - \mathbf{u}_h \quad \text{in} \quad \Omega\,, \\ \nabla\cdot\boldsymbol{\varphi} = 0 \quad \text{in} \quad \Omega\,, \\ \boldsymbol{\varphi} = 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{25}$$

belongs to $[H^2(\Omega)\cap H_0^1(\Omega)]^2 \times [H^1(\Omega)\cap L_0^2(\Omega)]$ and satisfies the following a priori estimate:

$$\|\boldsymbol{\varphi}\|_{2,\Omega} + \|\pi\|_{1,\Omega} \le C\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}. \tag{26}$$

Theorem 4 *Let us suppose that* $(\mathbf{u},p)$ *belongs to* $[H^{k+1}(\Omega)]^2 \times H^k(\Omega)$*, and the regularity hypotheses* (26) *. Then, there exists a constant* $C>0$*, independent of* h*, such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \le \begin{cases} C\,h^{k+1}\left(|\mathbf{u}|_{k+1,\Omega} + \|p\|_{k,\Omega}\right), & \text{if } Pe_K^2 < 1 \\ C\,h^{k+1/2}\left(|\mathbf{u}|_{k+1,\Omega} + \|p\|_{k,\Omega}\right), & \text{if } Pe_K^2 > 1. \end{cases} \tag{27}$$

Proof. - Multiplying the first equation on (25) by $\mathbf{u}-\mathbf{u}_h$, the second by $-(p-p_h)$, integrating by parts, additioning and using the consitency of the method we obtain

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2 &= \mathbf{B}\big((\mathbf{u}-\mathbf{u}_h, p-p_h),(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h, \pi-\pi_h)\big) \\ &+ \sum_{K\in\mathcal{T}_h}\big(\sigma(\mathbf{u}-\mathbf{u}_h) - \nu\Delta(\mathbf{u}-\mathbf{u}_h) + \mathbf{a}\cdot\nabla(\mathbf{u}-\mathbf{u}_h) + \nabla(p-p_h), \tau_K(\mathbf{u}-\mathbf{u}_h)\big)_K \end{aligned}$$

where $\boldsymbol{\varphi}_h \in \mathbf{V}_h$ and $\pi_h \in Q_h$ denote the Lagrange and Clément interpolate of $\boldsymbol{\varphi}$ and π, respectively. Then

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2 \le C &\left\{\sum_{K\in\mathcal{T}_h}\sigma\,\|\mathbf{u}-\mathbf{u}_h\|_{0,K}^2 + \nu\,|\mathbf{u}-\mathbf{u}_h|_{1,K}^2 + \|\mathbf{a}\cdot\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2\right. \\ &+ \|p-p_h\|_{0,K}^2 + \|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2 + \sigma^2\,\|\tau_K^{1/2}(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2 \\ &+ \nu^2\,\|\tau_K^{1/2}\Delta(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2 + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla(\mathbf{u}-\mathbf{u}_h) + \nabla(p-p_h))\|_{0,K}^2 \\ &\left.+ \|\delta_K^{1/2}\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2\right\}^{1/2} \\ &\left\{\sum_{K\in\mathcal{T}_h}\sigma\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,K}^2 + \nu\,|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h|_{1,K}^2 + \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,K}^2\right. \\ &+ \|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2 + \|\pi-\pi_h\|_{0,K}^2 + \sigma^2\,\|\tau_K^{1/2}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2 \\ &+ \nu^2\,\|\tau_K^{1/2}\Delta(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2 + \|\tau_K^{1/2}(\mathbf{a}\cdot\nabla(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) + \nabla(\pi-\pi_h))\|_{0,K}^2 \\ &\left.+ \|\delta_K^{1/2}\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2 + \|\tau_K^{1/2}(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2\right\}^{1/2}. \end{aligned}$$

Now, let $\tilde{\mathbf{u}}_h$ be the Lagrange interpolate of $\mathbf{u}$. Since $\tau_K \le C\,\frac{h_K^2}{\nu}$, using interpolation inequalities (20) and inverse inequality (12) we arrive at

$$\begin{aligned}\nu^2\,\|\tau_K^{1/2}\Delta(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2 &\le C\,\nu\,h_K^2\,\|\Delta(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2\\ &\le C\nu\,h_K^2\left(\|\Delta(\mathbf{u}-\tilde{\mathbf{u}}_h)\|_{0,K}^2 + \|\Delta(\tilde{\mathbf{u}}_h-\mathbf{u}_h)\|_{0,K}^2\right)\\ &\le C\nu\,h_K^{2k}\,|\mathbf{u}|_{k+1,K}^2 + C\nu\,|\mathbf{u}-\mathbf{u}_h|_{1,K}^2\,.\end{aligned}$$

Using the error estimate for $|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_h$ given in Theorem 3 we arrive at

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2 \le C(|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_h^2 + \nu h^{2k}\,|\mathbf{u}|_{k+1,\Omega}^2 + \nu\,|\mathbf{u}-\mathbf{u}_h|_{1,K}^2)^{1/2}\\ &\Big\{\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1,K}^2+\|\pi-\pi_h\|_{0,K}^2+\nu^2\|\tau_K^{1/2}\Delta(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2+\|\tau_K^{1/2}\mathbf{a}\cdot\nabla(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2\\ &\quad+\|\tau_K^{1/2}\nabla(\pi-\pi_h)\|_{0,K}^2+\|\delta_K^{1/2}\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2+\|\tau_K^{1/2}(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2\Big\}^{1/2}\\ &\le C\,h^k\big(|\mathbf{u}|_{k+1,\Omega}+\|p\|_{k,\Omega}\big)\\ &\Big\{\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1,K}^2+\|\pi-\pi_h\|_{0,K}^2+\nu^2\|\tau_K^{1/2}\Delta(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2+\|\tau_K^{1/2}\mathbf{a}\cdot\nabla(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2\\ &\quad+\|\tau_K^{1/2}\nabla(\pi-\pi_h)\|_{0,K}^2+\|\delta_K^{1/2}\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2+\|\tau_K^{1/2}(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2\Big\}^{1/2}\,.\end{aligned}$$

The rest of the proof is separated in two cases:
i). $Pe_K^2 < 1$ (i.e., $m_k|\mathbf{a}|_2 h_K < 4\nu$). In this case, we have

$$\tau_K = \frac{h_K^2}{\sigma h_K^2\xi(Pe_K^1)+\frac{4\nu}{m_k}} \le \frac{m_k h_K^2}{4\nu},\qquad \delta_K = \frac{\lambda|\mathbf{a}|_2^2 h_K^2 m_k}{4\nu}.$$

Hence, applying interpolation inequalities (20)-(21) and apriori estimate (26) we obtain

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2 &\le C\,h^k\big(|\mathbf{u}|_{k+1,\Omega}+\|p\|_{k,\Omega}\big)\\ &\Big\{\sum_{K\in\mathcal{T}_h}h_K^2\,\|\boldsymbol{\varphi}\|_{2,K}^2+h_K^2\,\|\pi\|_{1,K}^2+\nu\,h_K^2\,\|\Delta(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\|_{0,K}^2+\frac{\|\mathbf{a}\|_{\infty,\Omega}^2h_K^2}{\nu}\,|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h|_1^2\\ &\quad+\frac{h_K^2}{\nu}\,|\pi-\pi_h|_{1,K}^2+\frac{\lambda\|\mathbf{a}\|_{\infty,\Omega}h_K^2}{\nu}\,|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h|_{1,K}^2+\frac{h_K^2}{\nu}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,K}^2\Big\}^{1/2}\\ &\le C\,h^k\big(|\mathbf{u}|_{k+1,\Omega}+\|p\|_{k,\Omega}\big)\\ &\Big\{\sum_{K\in\mathcal{T}_h}h_K^2\,\|\boldsymbol{\varphi}\|_{2,K}^2+h_K^2\,\|\pi\|_{1,K}^2+\frac{h_K^2}{\nu}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,K}^2\Big\}^{1/2}\end{aligned}$$

$$\leq C\, h^{k+1}\, (|\mathbf{u}|_{k+1,\Omega} + \|p\|_{k,\Omega})\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}\,,$$

and the result follows.
ii). $Pe_K^2 > 1$ (i.e., $4\nu < m_k|\mathbf{a}|_2 h_K$). In this case, we get

$$\tau_K = \frac{h_K^2}{\sigma h_K^2 \xi(Pe_K^1) + |\mathbf{a}|_2 h_K} = \frac{h_K}{\sigma h_K + |\mathbf{a}|_2}\,,$$
$$\delta_K = \lambda |\mathbf{a}|_2 h_K\,.$$

Thus, supposing, without loss of generality, that $|\mathbf{a}(x)|_2 \geq a_* > 0$ in Ω, using interpolation inequalities and using the fact that $\tau_k \leq C\frac{h_K^2}{\nu}$ we get the following inequalities

$$\begin{aligned}
\nu^2\, \|\tau_K^{1/2} \Delta(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|_{0,K}^2 &\leq C\nu\, h_K^2\, \|\boldsymbol{\varphi}\|_{2,K}^2\,,\\
\|\tau_K^{1/2} \mathbf{a}\cdot \nabla(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|_{0,K}^2 &\leq \left\| \sqrt{\frac{h_K}{\sigma h_K + |\mathbf{a}|_2}}\, |\mathbf{a}|_2\, |\nabla(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)|_2 \right\|_{0,K}^2\\
&\leq C\, \|\mathbf{a}\|_{\infty,\Omega} h_K\, |\boldsymbol{\varphi} - \boldsymbol{\varphi}_h|_{1,\Omega}^2\\
&\leq C\, \|\mathbf{a}\|_{\infty,\Omega} h_K^3\, \|\boldsymbol{\varphi}\|_{2,K}^2\,,\\
\|\tau_K^{1/2} \nabla(\pi - \pi_h)\|_{0,K}^2 &\leq \frac{h_K}{\sigma h_K + a_*}\, |\pi - \pi_h|_{1,K}^2\\
&\leq \frac{h_K}{a_*}\, \|\pi\|_{1,K}^2\,,\\
\|\delta_K^{1/2} \nabla\cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|_{0,K}^2 &\leq \lambda \|\mathbf{a}\|_{\infty,\Omega} h_K\, |\boldsymbol{\varphi} - \boldsymbol{\varphi}_h|_{1,K}^2\\
&\leq \lambda \|\mathbf{a}\|_{\infty,\Omega}\, h_K^3\, \|\boldsymbol{\varphi}\|_{2,K}^2\,,\\
\|\tau_K^{1/2} (\mathbf{u} - \mathbf{u}_h)\|_{0,K}^2 &\leq \frac{h_K}{a_*}\, \|\mathbf{u} - \mathbf{u}_h\|_{0,K}^2\,.
\end{aligned}$$

The above inequalities lead to the final estimate

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 &\leq C\, h^k\, (|\mathbf{u}|_{k+1,\Omega} + \|p\|_{k,\Omega})\\
&\qquad \left\{ \sum_{K\in\mathcal{T}_h} h_K^2\, \|\boldsymbol{\varphi}\|_{2,K}^2 + h_K^2\, \|\pi\|_{1,K}^2 + \frac{h_K}{a_*}\, \|\pi\|_{1,K}^2 + \frac{h_K}{a_*}\, \|\mathbf{u} - \mathbf{u}_h\|_{0,K}^2 \right\}^{1/2}\\
&\leq C\, h^k\, \big(|\mathbf{u}|_{k+1,\Omega} + \|p\|_{k,\Omega}\big)\, \big\{ h\, \big(\|\boldsymbol{\varphi}\|_{2,\Omega}^2 + \|\pi\|_{1,\Omega}^2\big) + h\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 \big\}^{1/2}\\
&\leq C\, h^{k+1/2} \big(|\mathbf{u}|_{k+1,\Omega} + \|p\|_{k,\Omega}\big)\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}\,,
\end{aligned}$$

and the result follows. □

Remark 4 *The error analysis carried out in previous sections allows the constants to depend on the physical coefficientes σ, ν and $\mathbf{a}$. This is, of course, a problem from both the theoretical and numerical point of vue. Now, up to our knowledge, no general proof of error analysis independently of the physical coefficients is given for any stabilized finite element method. For a scalar reaction-difusion equation, in[15] there is an error analysis for the linear element case in which the authors give a constant which is independent of the physics, but if a higher order approximation is used, the proof is no longer valid. Due to these remarks, in the following section we perform numerical experiments tending to study the behavior of the error with respect to the physical constants, and show that our method is not dramatically affected by the limiting cases $\nu \to 0$ and $\sigma \to \infty$.* $\square$

3 NUMERICAL EXPERIMENTS

We use as domain the square $(0,1) \times (0,1)$, and we set $\mathbf{f}$ to be such as the exact solution of our problem (1) is given by

$$\begin{aligned} u_1(x_1,x_2) &= -256\,x_1^2(x_1-1)^2 x_2(x_2-1)(2x_2-1)\,, \\ u_2(x_1,x_2) &= -u_1(x_2,x_1)\,, \\ p(x_1,x_2) &= 150\,x_1(x_1-0.5)(x_2-0.5)\,. \end{aligned}$$

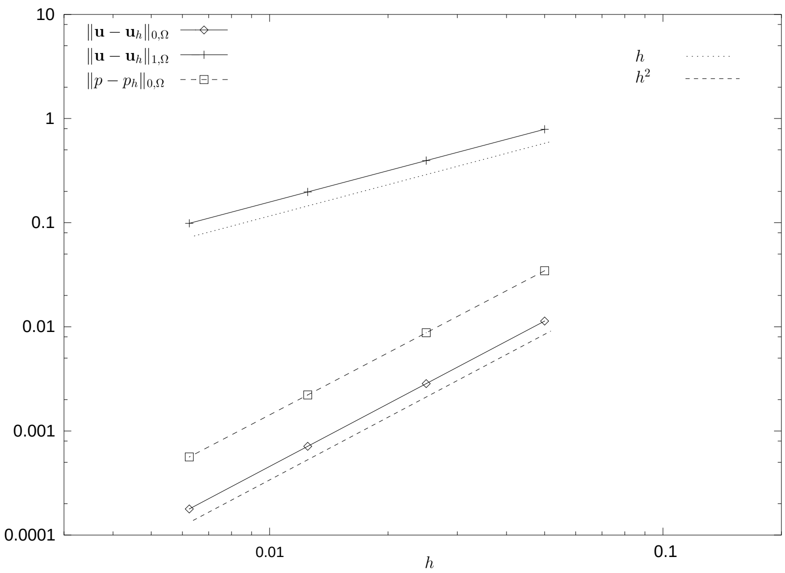

Figure 1: Convergence history: $\sigma = 1$, $\nu = 1$, $\mathbf{a} = (1,1)$

First of all, in[18] is shown that *div-div* term, $\sum_{K\in\mathcal{T}_h}(\nabla\cdot\mathbf{u}, \delta_K \nabla\cdot\mathbf{v})_K$, does not provide a significant improvement of the convergence rate. Because of this, in the sequel we will consider only $\lambda = 0$ in our computations. Using Q^1/Q^1 bilinear elements, we first report the diffusive dominated case with $\sigma = 1$, $\nu = 1$ and $\mathbf{a} = (1,1)$. The results is depicted in Figure 1, we

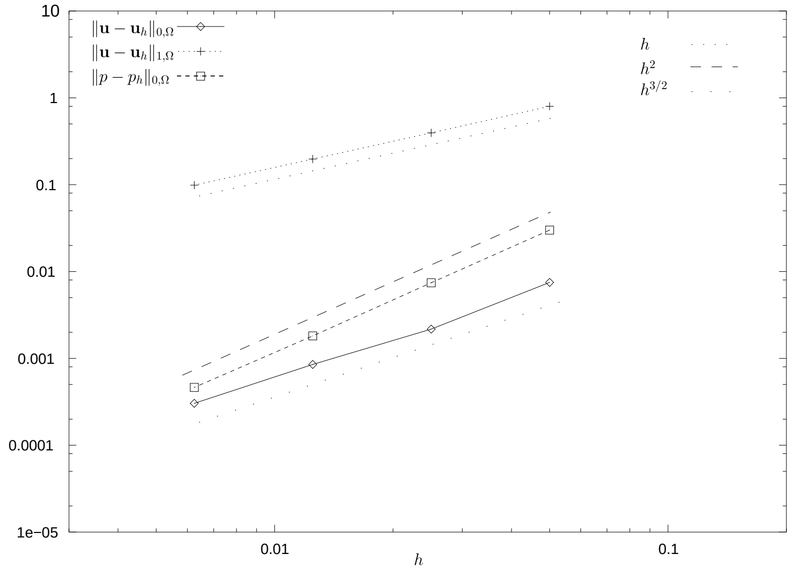

Figure 2: Convergence history: $\sigma = 100$, $\nu = 0.001$, $\mathbf{a} = (1, 1)$

recover optimal orders of convergence for velocity and pressure. In particular, we recover the h^2 order of convergence for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ which agrees with Theorem 4 since for this case $Pe_K^2 < 1$ even for the coarser mesh.

Next, we have considered the reaction-convection dominated case with $\sigma = 10^2, \nu = 10^{-3}$, $\mathbf{a} = (1, 1)$, see Figure 2 for the results. Since this case is convection dominated (indeed, we have $Pe_k^2 > 1$ even for the finer mesh), we recover the $h^{3/2}$ order of convergence on the $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ error.

To stress this fact in Figure 3 we show the convergence history for estimate $L^2(\Omega)$-norm with $\sigma = 100$, $\nu = 0.005$ and $\mathbf{a} = (1, 1)$, where we see that there is a zone in which the order is $h^{3/2}$ and when the mesh becomes fine enough (so that $Pe_K^2 < 1$) we recover the h^2 order.

σ	$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$	$\|p - p_h\|_{0,\Omega}$
0.1	2.7874×10^{-3}	0.3958	8.6038×10^{-3}
1	2.6803×10^{-3}	0.3957	8.4365×10^{-3}
10	2.3850×10^{-3}	0.3955	7.6346×10^{-3}
100	2.1733×10^{-3}	0.3958	7.4289×10^{-3}
10^3	2.1593×10^{-3}	0.3965	7.4812×10^{-3}
10^4	2.1606×10^{-3}	0.3968	7.4945×10^{-3}

Table 1: Behavior of the Finite Element error when σ grows

Now, we address the study of the sensitivity of the error to the physical coefficients. To this purpose, we use a uniform 40×40 mesh ($= 1600\ Q^1/Q^1$ elements), and we measure the errors in velocity and pressure.

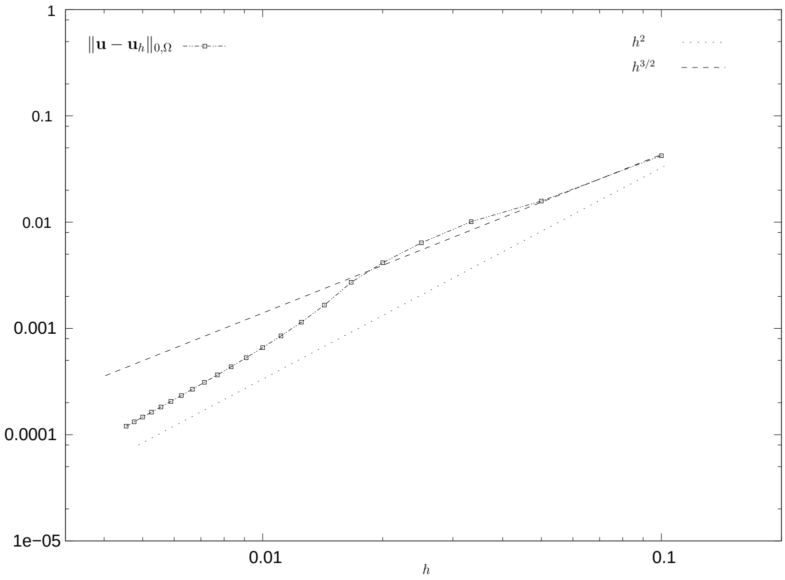

Figure 3: Convergence history: $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ with $\sigma = 100$, $\nu = 0.005$ and $\mathbf{a} = (1,1)$

ν	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$	$\|p-p_h\|_{0,\Omega}$
1	3.2727×10^{-3}	0.3950	1.00625×10^{-2}
0.1	3.6313×10^{-3}	0.3951	7.8391×10^{-3}
0.01	6.4479×10^{-3}	0.3981	6.6037×10^{-3}
10^{-3}	2.1733×10^{-3}	0.3958	7.4289×10^{-3}
10^{-4}	1.5185×10^{-3}	0.3985	7.9905×10^{-3}
10^{-5}	1.5045×10^{-3}	0.3990	8.0514×10^{-3}
10^{-6}	1.5038×10^{-3}	0.3990	8.0575×10^{-3}

Table 2: Behavior of the Finite Element error when ν decreases

$\mathbf{a}$	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$	$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$	$\|p-p_h\|_{0,\Omega}$
(0.1,0.1)	6.4927×10^{-3}	0.3987	7.8081×10^{-3}
(1,1)	2.1733×10^{-3}	0.3958	7.4289×10^{-3}
(5,5)	1.6817×10^{-3}	0.3980	8.8848×10^{-3}
(10,10)	1.7967×10^{-3}	0.3993	1.5086×10^{-2}
(20,20)	1.9944×10^{-3}	0.4011	3.6076×10^{-2}
(40,40)	2.2162×10^{-3}	0.4033	8.8978×10^{-2}

Table 3: Behavior of the Finite Element error when $|\mathbf{a}|$ grows

First, we fix $\nu = 10^{-3}$ and $\mathbf{a} = (1,1)$ and make σ grow. The results are shown in Table 1, where we see that the velocity error remains bounded while σ grows and that the pressure error presents a good behavior even for very large values of σ. Next, we fix $\sigma = 100$ and $\mathbf{a} = (1, 1)$, and make the viscosity ν decrease. The results are shown in Table 2, where we see that both errors are not singnificantly affected by the viscosity. Now, we fix $\sigma = 100$ and $\nu = 10^{-3}$, and let $|\mathbf{a}|$ grow. We observe, in Table 3, that the error in velocity remains bounded, while the error in pressure remains bounded for a quite large range of local Péclet numbers $\frac{m_k|\mathbf{a}|h_K}{4\nu}$. Numerical experiences beyond that range of Péclet numbers, have shown that the pressure error grows. This is reasonable since we are already dealing with relatively high Reynolds number.

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