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BEAM WITH RANDOM FIELD AND ENTROPY PROPAGATION

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Abstract. This paper investigates the impact of two probabilistic models of the flexural stiffness of a beam on its frequency response. The probabilistic models of the stochastic field are constructed using the maximum entropy principle, where different information is considered, such as bounds, mean value, etc. Each probabilistic model has a level of entropy; as the information given increases, the level of entropy of the model decreases. The response of the computational model of the beam is given in the frequency domain, and the entropy of the response is analyzed. Doing so, it is possible to quantify the propagation of the information, given the costruction of the probabilistic model, measured by Shannon's entropy, throughout the system.

1 INTRODUCTION

This paper investigates the random dynamical response of a beam with uncertain flexural stiffness. This uncertainty is relative to the variability appearing during the manufacturing process or during the cycle life (uncontrolled damages) of the structure. The propagation of the uncertainty throughout the response of the beam is analyzed in terms of Shannon's Entropy (Shannon, 1948).

Usually, the propagation of uncertainties is analyzed through the observation of the input/output variance (Arnoux et al., 2013; Ritto et al., 2009). The main drawback with such a methodology is that some distribution with few uncertainty can have a large variance (for instance, a bi-modal distribution for which the distance between the two peaks is large), then the variance might not be reliable measure of uncertainty. For this reason entropy-based sensitivity indexes have been introduced in H. Liu (2005) as an alternative to the classical Sobol indexes (Shannon, 1993) which are variance-based.

The first step consists in constructing a probabilistic model for these parameters, i.e., construct the probability density functions (pdf) for the random variables modeling the uncertain parameters. The Shannon's entropy measures the relative uncertainty associated to a pdf. The Maximum Entropy (MaxEnt) principle (Shannon, 1948; Jaynes, 1994; Kapur and Kevasan, 1992) is a powerful method which allows the pdf of a random variable to be constructed from a set of available information. This method consists in choosing the pdf which maximizes the entropy (and thus the uncertainty) under the constraints defined by the available information. Therefore, the entropy (uncertainty) level in the input parameters depends on the amount of information available for the uncertain parameters. It is then interesting to analyze how this level of entropy propagates into the quantities of interest.

This paper is organized as follows. In Section 2, the deterministic model is presented. Then, Section 3 is devoted to the construction of the stochastic computational model. Finally, in Section 4, the numerical results are analyzed. The conclusions are made in Section 5.

2 NOMINAL COMPUTATIONAL MODEL

In this section the reduced nominal computational model is constructed using the Finite Element method and the model reduction is performed using a classical modal analysis.

Considering a homogeneous Euler-Bernoulli beam, the partial differential equation governing the dynamics of the structure is written as:

$$m\frac{\partial^2 v(x,t)}{\partial t^2} + EI\frac{\partial^4 v(x,t)}{\partial x^4} = f(x,t) \qquad x \in [0,L]\,,\, t \in [0,T]\,, \tag{1}$$

where v is the transversal displacement, L is the length of the beam, m is the mass per unit length, E is the elasticity modulus, I is the area moment of inertia and f is the external force per unit length.

Let $v = \hat{v}\exp(i\omega t)$ and $f = \hat{f}\exp(i\omega t)$, in which the hat means the amplitude in a given frequency ω (steady state response), and $i = \sqrt{-1}$. Substituting $v = \hat{v}\exp(i\omega t)$ and $f = \hat{f}\exp(i\omega t)$ in Eq.(1) leads to:

$$-\omega^2 m\hat{v} + EI\frac{\partial^4 \hat{v}}{\partial x^4} = \hat{f}\,. \tag{2}$$

The partial differential equation, Eq. (2), is discretized by means of the Finite Element method: $\hat{v}^{(e)}(\xi,\omega) = \mathbf{N}(\xi)^T\hat{\mathbf{v}}^{(e)}(\omega)$, in which $\mathbf{N}$ are the shape functions (Hermitian functions), ξ is the element coordinate, and $\hat{\mathbf{v}}^{(e)}$ is the vector with the element displacements.

After assembling the element matrices, and including a proportional damping matrix, the discretized system is given by:

$$-\omega^2[M]\hat{\mathbf{v}}(\omega) + i\omega[C]\hat{\mathbf{v}}(\omega) + [K]\hat{\mathbf{v}}(\omega) = \hat{\mathbf{f}}(\omega)\,, \tag{3}$$

where $[M]$, $[K]$ are the mass and stiffness matrices, and $[C] = \alpha[M] + \beta[K] \in \mathbb{R}^{m\times m}$, $\hat{\mathbf{v}}(\omega)$ $\in \mathbb{C}^m$ is the response vector and $\hat{\mathbf{f}}(\omega) \in \mathbb{C}^m$ is the force vector.

We are interested in the frequency response of the structure on the frequency band of analysis $B = [0, \omega_{max}]$. For all $\omega \in B$, the vector $\mathbf{v}(\omega)$ is the solution of the following matrix equation

$$(-\omega^2[M] + i\omega[C(\mathbf{h})] + [K(\mathbf{h})])\,\mathbf{v}(\omega) = \hat{\mathbf{f}}\,(\omega)\,, \tag{4}$$

in which $\mathbf{h}$ is the set of parameters that will later, in the next Section, be modeled with random variables.

The reduced nominal computation model is constructed using the modal analysis reduction method. Let $\mathcal{C}_h$ be the admissible set for the vector $\mathbf{h}$. Then for all $\mathbf{h}$ in $\mathcal{C}_h$, the n first eigenvalues $0 < \lambda_1(\mathbf{h}) \leq \lambda_2(\mathbf{h}) \leq \ldots \leq \lambda_n(\mathbf{h})$ associated with the elastic modes $\{\phi_1(\mathbf{h}), \phi_2(\mathbf{h}), \ldots, \phi_n(\mathbf{h})\}$ are solutions of the following generalized eigenvalue problem

$$[K(\mathbf{h})]\,\phi(\mathbf{h}) = \lambda(\mathbf{h})[M]\,\phi(\mathbf{h})\,. \tag{5}$$

The reduced-order nominal computation model is obtained by projecting the nominal computation model on the subspace spanned by the n first elastic modes calculated using Eq. (5). Let $[\Phi(\mathbf{h})]$ be the $m \times n$ matrix whose columns are the n first elastic modes. We then introduce the approximation

$$\hat{\mathbf{v}}(\omega) = [\Phi(\mathbf{h})]\,\mathbf{q}(\omega)\,, \tag{6}$$

in which the vector $\mathbf{q}(\omega)$ is the vector of the n generalized coordinates and is the solution of the following reduced matrix equation

$$(-\omega^2[M_r(\mathbf{h})] + i\omega[C_r(\mathbf{h})] + [K_r(\mathbf{h})])\,\mathbf{q}(\omega) = [\Phi(\mathbf{h})]^T\hat{\mathbf{f}}(\omega)\,, \tag{7}$$

in which $[M_r(\mathbf{h})] = [\Phi(\mathbf{h})]^T\,[M]\,[\Phi(\mathbf{h})]$, $[C_r(\mathbf{h})] = [\Phi(\mathbf{h})]^T\,[C(\mathbf{h})]\,[\Phi(\mathbf{h})]$ and $[K_r(\mathbf{h})] = [\Phi(\mathbf{h})]^T\,[K(\mathbf{h})]\,[\Phi(\mathbf{h})]$ are the $n \times n$ mass, damping and stiffness reduced matrices.

3 STOCHASTIC COMPUTATIONAL MODEL

In this section, the stochastic computational model is derived from the reduced nominal computational model introduced in the previous section. The uncertain parameter field of the dynamical system considered is the bending stiffness $EI(x)$. It is modeled by a random field, which means that the field $\{\eqsim(\mathbf{x}), \mathbf{x} \in \Omega\}$ is modeled by the random field $\{\mathbb{H}(\mathbf{x}), \mathbf{x} \in \Omega\}$. Furthermore, it is assumed that the random field $\{\mathbb{H}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is homogeneous. Therefore, in the context of the FE discretization introduced in the previous section, the vector $\mathbf{h}$ which corresponds to the spatial discretization of $\eqsim(\mathbf{x})$, is modeled by a random vector $\mathbf{H}$. It should be remarked that the bending stiffness $EI(x)$ is constant along each element and, therefore, piecewise discontinuous. The probabilistic model of this random vector is constructed using the MaxEnt principle (Shannon, 1948; Jaynes, 1994; Kapur and Kevasan, 1992). Finally, the stochastic reduced-order computational model is presented.

In the present analysis the random field is considered independent, which means that for $(\mathbf{x}, \mathbf{x}')$ in Ω^2, $\eqsim(\mathbf{x})$ and $\eqsim(\mathbf{x}')$ are independent random variable. Then the available information is introduced independently for each component of the random vector $\mathbf{H}$. Since it is assumed that the random field $\{\mathbb{H}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is homogeneous, the available information is the same for all the components of the random vector $\mathbf{H}$. The support of the pdf of each component H_i is denoted by $\mathcal{K}$ such that $\mathcal{K} \subset \mathbb{R}$. Let $E\{.\}$ be the mathematical expectation. For each component, H_i ($i = 1, \ldots, N$), the available information is written as

$$E\{\mathbf{g}(H_i)\} = \mathbf{f}_i \,, \tag{8}$$

in which $h \mapsto \mathbf{g}(h)$ is a given function from $\mathbb{R}$ into $\mathbb{R}^\mu$ and where $\mathbf{f}_i$ is a given function in $\mathbb{R}^\mu$. Equation (8) can be rewritten as

$$\int_{\mathbb{R}} \mathbf{g}(h_i) p_{h_i}(h_i) \mathrm{d}h_i = \mathbf{f}_i \,. \tag{9}$$

An additional constraint relative to the normalization of the joint pdf $p_{\mathbf{H}}(\mathbf{h})$ is introduced such that

$$\int_{\mathbb{R}^N} p_{\mathbf{H}}(\mathbf{h}) \mathbf{dh} = 1 \,. \tag{10}$$

The entropy of the joint pdf $\mathbf{h} \mapsto p_{\mathbf{H}}(\mathbf{h})$ is defined by

$$S(p_{\mathbf{H}}) = -\int_{\mathbb{R}^N} p_{\mathbf{H}}(\mathbf{h}) \, \log(p_{\mathbf{H}}(\mathbf{h})) \mathbf{dh} \,, \tag{11}$$

where $\log$ is the Neperian logarithm. This functional measures the uncertainty for $p_{\mathbf{H}}$. Let $\mathcal{C}$ be the set of all the pdf defined on $\mathbb{R}^N$ with values in $\mathbb{R}^+$, verifying the constraints defined by Eqs. (9) and (10). Then the MaxEnt principle consists in constructing the probability density function $\boldsymbol{h} \mapsto p_{\mathbf{H}}(\mathbf{h})$ as the unique pdf in $\mathcal{C}$ which maximizes the entropy $S(p_{\mathbf{H}})$. Then by introducing a Lagrange multiplier λ_0 in $\mathbb{R}^+$ associated with Eq. (10) and N Lagrange multipliers $\boldsymbol{\lambda}_i$ associated with Eq. (9) and belonging to an admissible open subset $\mathcal{L}_\mu$ of $\mathbb{R}^\mu$, it can be shown (see Jaynes (1994); Kapur and Kevasan (1992)) that the MaxEnt solution, if it exists, is defined by

$$p_{\mathbf{H}}(\mathbf{h}) = \prod_{i=1}^{N} \{\mathbb{1}_{\mathcal{K}}(h_i)\} \, c_0^{\mathrm{sol}} \, \exp(-\sum_{i=1}^{N} \langle \boldsymbol{\lambda}_i^{\mathrm{sol}}, \mathbf{g}(h_i)\rangle) \,, \tag{12}$$

in which the indicator function $h_i \mapsto \mathbb{1}_{\mathcal{K}}(h_i)$ is such that it is equal to 1 if $h_i \in \mathcal{K}$ and is zero otherwise. In Eq. (12), $c_0^{\mathrm{sol}} = \exp(-\lambda_0^{\mathrm{sol}})$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1 y_1 + \ldots + x_\mu y_\mu$ and λ_0^{sol} and $\boldsymbol{\lambda}_i^{\mathrm{sol}}$ are respectively the values of λ_0 and $\boldsymbol{\lambda}_i$ for which Eqs. (9) and (10) are satisfied. Equation (12) shows that the components h_i of the the random vector $\mathbf{H}$ are independent random variables for which the pdfs are given for i in $\{1, \ldots, N\}$ by

$$p_{H_i}(h_i) = \mathbb{1}_{\mathcal{K}}(h_i) \, c_i^{\mathrm{sol}} \, \exp(-\langle \boldsymbol{\lambda}_i^{\mathrm{sol}}, \mathbf{g}(h_i)\rangle) \,, \tag{13}$$

Using the normalization condition, the parameter c_i^{sol} can be eliminated, and Eq. (13) can be rewritten as

$$p_{H_i}(h_i) = \mathbb{1}_{\mathcal{K}}(h_i) \, c_i(\boldsymbol{\lambda}_i^{\mathrm{sol}}) \, \exp(-\langle \boldsymbol{\lambda}_i^{\mathrm{sol}}, \mathbf{g}(h_i)\rangle) \,, \tag{14}$$

in which $c_i(\boldsymbol{\lambda_i})$ is defined by

$$c_i(\boldsymbol{\lambda_i}) = \left\{ \int_{\mathcal{K}} \exp(-\langle \boldsymbol{\lambda}_i, \boldsymbol{g}(h_i) \rangle) \, \mathrm{d}h_i \right\}^{-1} . \tag{15}$$

The N Lagrange multipliers $\boldsymbol{\lambda}_i$ are then calculated using Eqs. (9), (14) and (15). The integrals can be calculated explicitly for some particular cases of available information. Since, the dimension of these integrals is one, they can be calculated using any numerical integration method.

If we consider that the support $\mathcal{K}$ is compact, and no more information is introduced, the MaxEnt distribution is the uniform distribution. Then**,** if information is added, a constraint is added and the research manifold for the maximum of the entropy is reduced yielding a smaller maximum. More specifically, if we introduce the two random variables H_1 and H_2, with the same support, for which the available information are respectively defined by

$$E\{\boldsymbol{g}^1(H_1)\} = \boldsymbol{f}^1 , \tag{16}$$

$$E\{\boldsymbol{g}^1(H_2)\} = \boldsymbol{f}^1 , \; E\{\boldsymbol{g}^2(H_2)\} = \boldsymbol{f}^2 , \tag{17}$$

in which the functions $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ are independent. Then, the available information relative to H_1 is included in the available information relative to H_2. Let S_1 be the maximum entropy relative to H_1 with the constraint defined by Eq. (16) and S_2 be the maximum entropy relative to H_2 with the constraint defined by Eq. (17). We then have

$$S_1 \geq S_2 . \tag{18}$$

Proceeding in this manner, it is possible to create nested probabilistic models with increasing information, and hence decreasing entropy.

The stochastic computational model is derived from the reduced nominal computational model introduced in the last Section, for which the deterministic vector $\mathbf{h}$ of the discretization of the uncertain fields is replaced by the random vector $\mathbf{H}$. The n first random eigenvalues $0 < \Lambda_1(\mathbf{H}) \leq \ldots \leq \Lambda_n(\mathbf{H})$ associated with the random elastic modes $\{\psi_1(\mathbf{H}), \ldots, \psi_n(\mathbf{H})\}$ are solutions of the following random eigenvalue problem

$$[\mathbb{K}(\mathbf{H})] \, \psi(\mathbf{H}) = \Lambda(\mathbf{H})[M] \, \psi(\mathbf{H}) . \tag{19}$$

Then for all ω in B, the random response $\mathbf{V}(\omega)$ of the stochastic reduced-order computational model, is written as

$$\mathbf{V}(\omega) = [\Psi(\mathbf{H})] \, \mathbf{Q}(\omega) , \tag{20}$$

in which the random vector $\mathbf{Q}(\omega)$ of the random generalized coordinates, is the solution of the following random reduced-order matrix equation,

$$(-\omega^2[\mathbb{M}_r(\mathbf{H})] + i\omega[\mathbb{C}_r(\mathbf{H})] + [\mathbb{K}_r(\mathbf{H})]) \, \mathbf{Q}(\omega) = [\Psi(\mathbf{H})]^T \mathbf{f} . \tag{21}$$

in which $[\mathbb{M}_r(\mathbf{H})] = [\Psi(\mathbf{H})]^T \, [M] \, [\Psi(\mathbf{H})]$, $[\mathbb{C}_r(\mathbf{H})] = [\Psi(\mathbf{H})]^T \, [\mathbb{C}(\mathbf{H})] \, [\Psi(\mathbf{H})]$ and $[\mathbb{K}_r(\mathbf{H})] = [\Psi(\mathbf{H})]^T \, [\mathbb{K}(\mathbf{H})] \, [\Psi(\mathbf{H})]$ are the random $n \times n$ mass, damping and stiffness reduced matrices. This equation can be solved using the Monte Carlo simulation method (Rubinstein and Kroese, 2007).

4 NUMERICAL RESULTS

A clamped-clamped beam is considered in the present analysis, with length $L = 1$ m, diameter $D = 0.01$ m, cross sectional area $A = \pi D^2/4$, Elasticity Modulus $E = 200$ GPa and mass density $\rho = 7850$ kg/m^3. Fifteen normal modes are used in the reduced-order model and the frequency band analyzed is [0,3000] Hz. The observation is taken at $x = 0.41$ m and the velocity spectrum of the response is analyzed.

Two input stochastic model are analyzed for $\mathbf{H}$: (1) independent Uniform, where only a compact support is given as information for the construction of the probabilistic model, $[\mathrm{H}_{\min}, \mathrm{H}_{\max}]$, and (2) independent Beta, where, besides the compact support, two more constraints are considered: (a) $E\{\log(H_i - \mathrm{H}_{\min})\} = \mathrm{c}_1 < +\infty$ and (b) $E\{\log(\mathrm{H}_{\max} - H_i)\} = \mathrm{c}_2 < +\infty$, which means that the likelihood of the random variables reaches zero near the boundaries.

Figure 1 shows the response of the nominal model at $x = 0.41$ m. The velocity spectrum show some peaks corresponding to the natural frequencies of the structure.

Let us consider plus or minus 25% of the mean bending stiffness as the compact support for the random Uniform field, $[0.75EI, 1.25EI]$, and the Beta parameters $\alpha = 2$ and $\beta = 2$. Figure 2 shows the stochastic response in the frequency domain. The mean response is plotted together with the 98% confidence envelope. The results are quite similar for the two models. Figure 3 shows the coefficient of variation (standard deviation over the mean) at each frequency. It is noted that the uncertainty is greater close to the resonance and anti-resonance frequencies of the structure.

The level of the uncertainty measured by Shannon's entropy can easily be computed for the random bending stiffness at each x: $S_{unif} = 3.8936$ for the Uniform pdf and $S_{beta} = 3.7684$ for the Beta pdf. It is observed that the entropy related to the Uniform field is bigger than the entropy related to the Beta field, as expected. To better compare the response entropy with the coefficient of variation, the marginal entropies of the response at each frequency is computed. Figure 4 shows the entropy of the marginal pdf of the response. Note that the level of uncertainty of the response for the Uniform pdf is bigger than the response for the Beta pdf. Also as expected, the behavior of the Entropy is similar to the behavior of the coefficient of variation, since both measure the uncertainty of the response. Nevertheless, the curves in Figures 3 and 4 are different. Finally, Figure 5 shows the histogram of the random variable related to the response at frequency (860 Hz): in the present analysis the output pdf is uni-modal.

It is concluded that the entropy of the response can be used as an alternative analysis of uncertainty propagation in random structural dynamics.

5 CONCLUDING REMARKS

The random response of a linear clamped-clamped Euler-Bernoulli beam with uncertain bending stiffness was analyzed. Two random fields related to the bending stiffness were constructed using the MaxEnt principle with different information. The uncertainty propagation throughout the system was analyzed in terms of Shannon's entropy measure. It is argued that the propagation viewed in terms of entropy is well suited for the problem. One advantage of the entropy measure is that it is more appropriate than the variance in some cases, such as a bi-modal distribution.

An ongoing work is been performed to construct correlated random input random fields, and their propagation throughout the system.

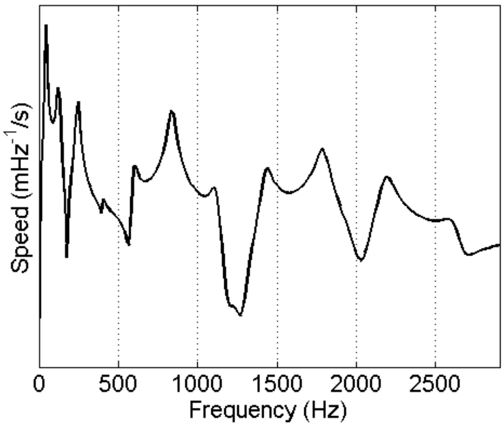
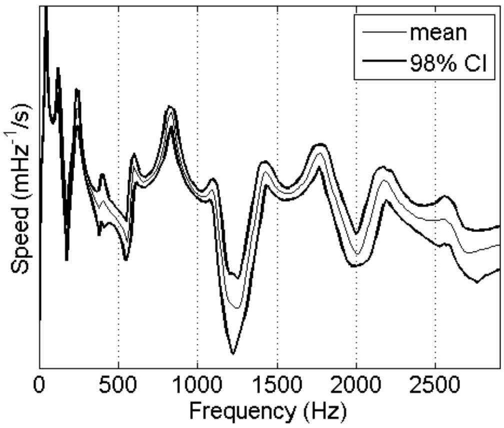
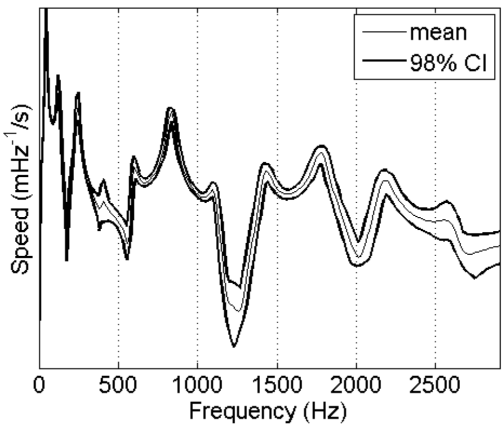


Figure 1: Spectrum of velocity of the nominal model at $x = 0.41$ m.



(a)



(b)

Figure 2: Mean and 98% confidence limits of the response at $x = 0.41$ m: (a) Uniform input and (b) Beta input.

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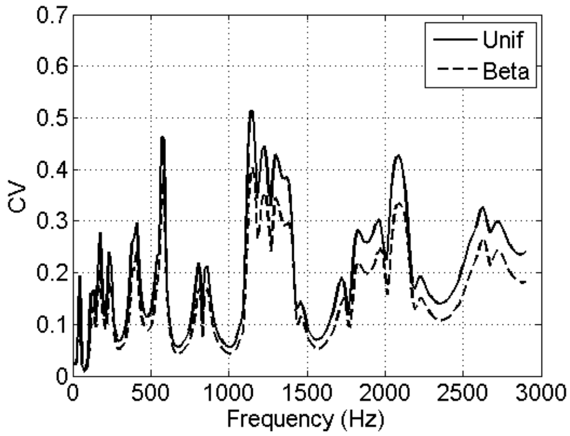

Figure 3: Coefficient of variation of the response for two different input random fields.

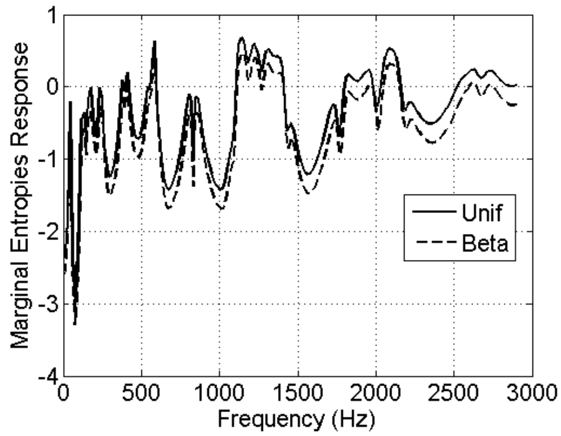

Figure 4: Marginal entropies of the response for two different input random fields.

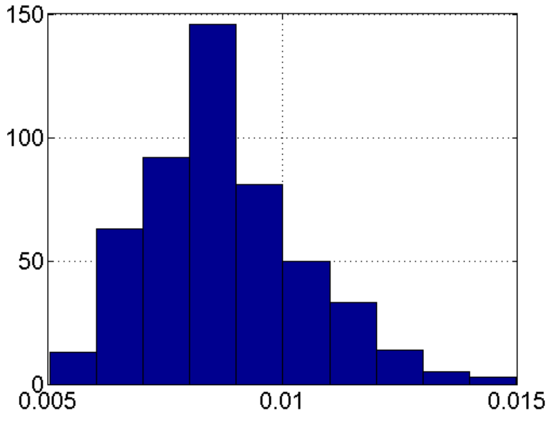

Figure 5: Histogram of the speed at 860 Hz.

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