

## A NEW MULTI-OBJECTIVE OPTIMIZATION ALGORITHM FOR NONCONVEX PARETO FRONTS AND OBJECTIVE FUNCTIONS

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**Keywords:** Multi-objective optimization, normal boundary intersection, global optimization, probabilistic restart.

**Abstract.** The main goal of a multi-objective optimization problem (MOP) is to construct the Pareto front (PF), which represents the best tradeoffs among the objectives to be optimized. The development of MOP methods has sought the generation of an even spread of Pareto optimal points as well as the treatment of the non-convexity on both the Pareto front and the functions to be minimized themselves. Among other approaches, the normal boundary intersection (NBI) is able to deal with nonconvex PF and it also provides an even spread of Pareto optimal points. To deal with the non-convexity of the functions to be minimized some researchers have employed heuristic algorithms in the construction of the Pareto Front. However, the computational cost associated to heuristics is very high and the application of these methods to expensive functions (e.g. structures modeled by a finite element method) may become nonviable. Another difficulty arises in coupling a heuristic algorithm and the NBI approach: the latter imposes an equality constraint and the heuristic algorithm must then employ a penalty method to handle such a constraint, which usually worsens the performance of the algorithm. Clearly, the main issue for the utilization of heuristic algorithms is the computational cost they require. With the limitations of heuristic algorithms in mind, we propose a new algorithm for the solution of MOP whose PF and the functions to be minimized are nonconvex. It applies the NBI to discretize the PF and it employs a global optimization algorithm based on a probabilistic restart procedure and local searches to minimize each NBI subproblem. The accuracy and efficiency of the proposed algorithm is shown in the numerical section comparing its results to well-known heuristic algorithms.

## 1 INTRODUCTION

A great number of real life optimization problems require that two or more characteristics (or performance function) of the system under analysis be optimized simultaneously. Normally, these performance functions conflict, i.e. it is not possible to find a single design that minimizes all functions. Thus, the main goal of a multi-objective optimization problem (MOP) is to construct the Pareto front (PF), which represents the best tradeoffs among the objectives to be optimized.

The PF may be approximated as the solution of a series of scalar optimization subproblems in which the objective is an aggregation of all the objective functions. Several methods are found in the literature for constructing aggregation functions, such as the weighted sum, Tchebycheff inequality, the normal boundary intersection, the normal constraint method, the Physical Programming method, Goal Programming, the epsilon constraints and Directed Search Domain (Messac (1996); Das and Dennis (1998); Deb (2001); Messac et al. (2003); Messac and Mattson (2003); Zhang and Li (2007); Chinchuluun and Pardalos (2007); Mueller-Gritschneider et al. (2009)).

The development of MOP methods has sought the generation of an even spread of Pareto optimal points as well as the treatment of the non-convexity on both the Pareto front and the functions to be minimized themselves. Among other approaches, the NBI is able to deal with nonconvex PF and it also provides an even spread of Pareto optimal points (Das and Dennis (1998)). To deal with these non-convexities, some researchers have employed heuristic algorithms in the construction of the Pareto Front (Zhang and Li (2007)). However, the computational cost associated to heuristics is very high and the application of these methods to expensive functions (e.g. structures modeled by a finite element method) may become nonviable. Another difficulty arises in coupling a heuristic algorithm and the NBI approach: the latter imposes an equality constraint and the heuristic algorithm must then employ a penalty method to handle such a constraint, which usually worsens the performance of the algorithm.

Clearly, the main issue for the utilization of heuristic algorithms is the computational cost they require. With the limitations of heuristic algorithms in mind, we propose a new algorithm for the solution of MOP able to handle nonconvex PF and nonconvex objective functions. In order to achieve this goal it applies the NBI to discretize the PF and it employs a global optimization algorithm based on a restart procedure and local searches to minimize each NBI subproblem. The original version of this restart procedure was developed by Luersen and Riche (2004) and its coupling with gradient based algorithms was tested by Torii et al. (2011).

This paper is organized as follows. The multi-objective optimization problem is posed in Section 2. The proposed algorithm is detailed in Section 3. Finally, the numerical results are presented in Section 4 and the concluding remarks are given in Section 5.

## 2 MULTI-OBJECTIVE OPTIMIZATION (MOP)

A general MOP may be stated as:

$$\underline{\mathbf{s}}^* = \arg \min_{\underline{\mathbf{s}} \in \mathcal{C}} \mathbf{J}(\underline{\mathbf{s}}) = (J_1(\underline{\mathbf{s}}), \dots, J_m(\underline{\mathbf{s}})), \quad (1)$$

where  $\underline{\mathbf{s}}$  is the design vector,  $\mathcal{C}$  is the decision space,  $\mathbf{J} : \mathcal{C} \mapsto \mathcal{J} \subset \mathcal{R}^m$  consists of  $m$  real-valued objective functions and  $\mathcal{J}$  is called the objective space. Usually, the objectives contradict each other and, consequently, no point in  $\mathcal{C}$  minimizes all of them simultaneously. The best tradeoffs among the objectives are defined in terms of Pareto optimality, and the main

goal of the MOP is to construct the PF. In the next section, we present the proposed MOP algorithm.

### 3 OPTIMIZATION ALGORITHM

In order to solve the MOP stated in the last section, we employ the NBI approach to decompose the PF into a set of scalar optimization subproblems. In order to solve these subproblems, we take advantage of the efficiency of gradient based algorithms (especially to deal with the equality constraint imposed by the NBI) and we couple it with a probabilistic restart in order to avoid local minima. This procedure is detailed in the sequel.

#### 3.1 Normal Boundary Intersection

Two important features of the NBI method are (Das and Dennis (1998)): (i) it produces an even spread of points on the PF and, (ii) it can also handle problems where the Pareto surface is discontinuous or non-smooth. NBI works by transforming the non-linear MOP into a set of nonlinear scalar subproblems. This strategy can be considered as the state of the art regarding deterministic methods. It uses a geometrically intuitive parametrization to produce a set of points on the PF, giving an accurate picture of the whole surface.

In a MOP, we have  $m$  objective functions which we want to minimize simultaneously. Since some of the objective functions may conflict with others, one has to find an appropriate compromise or trade-off among these  $m$  objectives. The ideal situation would be the existence of a vector  $\underline{s}^*$  that minimizes the  $m$  objective functions simultaneously such as:

$$(J_1(\underline{s}^*), \dots, J_m(\underline{s}^*)) = (J_1^*, \dots, J_m^*)$$

where each  $J_i^*$ ,  $i = 1, \dots, m$ , is the individual minima of the corresponding scalar problem

$$\underline{s}^* = \arg \min_{\underline{s} \in \mathcal{C}} J_i(\underline{s}) \quad (2)$$

for  $i = 1, \dots, m$ . However, this is an ideal situation and normally the individual minimum  $J_i^*$  will be attained at different points, i.e. usually each scalar problem  $i$  of Eq.(2) has a different solution  $\underline{s}_i^*$ . NBI essentially works by solving sequentially a set of single nonlinear problems (NBI subproblems), which are defined as:

$$\begin{cases} \underline{s}^* = \arg \max_{\underline{s} \in \mathcal{C}} l(\underline{s}) \\ \text{subject to } \Phi \mathbf{w} + l \hat{\mathbf{n}} = \mathbf{J}(\underline{s}) - \mathbf{J}^*, \end{cases} \quad (3)$$

Let  $\Phi$  be the  $m \times m$  pay-off matrix in which the  $i^{th}$  column is  $\mathbf{J}(\underline{s}_i^*) - \mathbf{J}^*$ , where  $\mathbf{J}^*$  is the vector of the individual minima (i.e., the utopia point or shadow minimum) and  $\underline{s}_i^*$  is the minimizer of objective  $J_i$  (i.e., minimizers of problem (2)).  $\mathbf{w}$  is a vector of weights such that  $\sum_{i=1}^m w_i = 1$ ,  $w_i \geq 0$ , and  $\hat{\mathbf{n}}$  is the quasi-normal direction pointing towards the origin of the objective space  $\mathcal{J} = \{\mathbf{J}(\underline{s}) : \underline{s} \in \mathcal{C}\}$ .  $\Phi \mathbf{w}$  defines a point on the so-called Convex Hull of Individual Minima (CHIM) in the objective space  $\mathcal{J}$ , i.e. the CHIM is the set of points that are convex combinations of  $\mathbf{J}(\underline{s}_i^*) - \mathbf{J}^*$ . Figure 1 illustrates the aforementioned quantities for  $m = 2$ . In this figure, the shaded area is the objective space  $\mathcal{J}$ , A and B are the individual minima  $\mathbf{J}(\underline{s}_2^*)$  and  $\mathbf{J}(\underline{s}_1^*)$ , respectively, and the dotted line connecting points A and B is the CHIM. Also, the PF is the curve ACB.

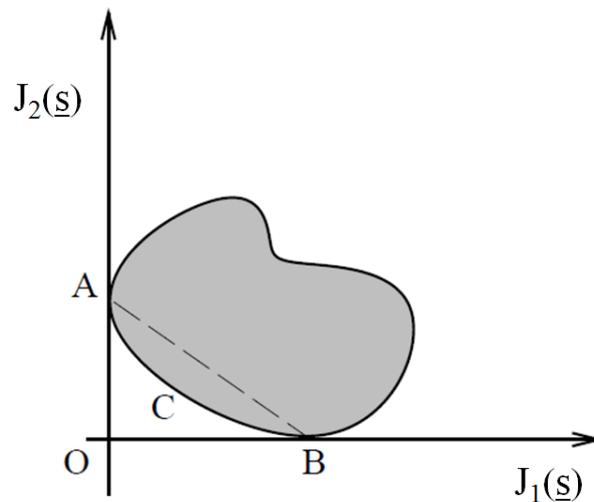


Figure 1: Objective space, individual minima and CHIM.

Now let us illustrate algebraically how the optimization problem given by Eq.(3) may provide us a point on the PF, i.e. a point on the boundary of  $\mathcal{J}$  (on the curve ACB). Given the weights  $\mathbf{w}$ ,  $\Phi\mathbf{w}$  represents a point in the CHIM. Let  $\hat{\mathbf{n}}$  denote the unit normal to the CHIM simplex pointing towards the origin, then  $\Phi\mathbf{w} + l\hat{\mathbf{n}}$ ,  $l \in \mathbb{R}$  represents the set of points on the normal. Thus, the intersection between the normal to the CHIM and the boundary of the objective space  $\mathcal{J}$  closest to the origin is expected to be Pareto-optimal point. Subproblem (3) is solved for different values of  $\mathbf{w}$ , so that an equally distributed set of them produces an equally distributed set of Pareto points. If the PF is convex and the individual minima of the objective are the global ones, the solution of each subproblem is a Pareto point.

The subproblem (3) shall be referred to as the *NBI* subproblem and written as  $NBI_{\mathbf{w}}$ , since  $\mathbf{w}$  is the characterizing parameter of the subproblem. The solution of these subproblems will be referred to as *NBI* points. The idea is to solve  $NBI_{\mathbf{w}}$  for various  $\mathbf{w}$  and find several points on the boundary of  $\mathcal{J}$ , effectively constructing a pointwise approximation of the PF. The number of points chosen to discretize the PF is set by the design and here it is given by the variable  $n_{sub}$ . Unfortunately, a point generated by NBI may not be a Pareto point if the boundary of the attained set in the objective space  $\mathcal{J}$  containing the Pareto points is non-convex (Das and Dennis (1998)). Although most of the applications found in the literature have convex PF, it is not the case of the MOP under analysis as it is shown in section 4. To eliminate the NBI points which are not Pareto points from the obtained PF, we verify whether the NBI point is a non-dominated point or not. The concept of non-domination or Pareto optimality is the basis of the MOP: a point  $\underline{\mathbf{s}}^* \in \mathcal{C}$  is said to be Pareto optimal or non-dominated point if and only if does not exist  $\underline{\mathbf{s}} \in \mathcal{C}$  satisfying  $\mathbf{J}(\underline{\mathbf{s}}) \prec \mathbf{J}(\underline{\mathbf{s}}^*)$ .

### 3.2 Probabilistic restart

It is common in engineering problems to find nonconvex function. If a given objective function is not convex, the NBI subproblems described in the section above may be nonconvex. Under this conditions, deterministic optimization algorithms such as gradient methods, Newton methods, or sequential simplex methods, may not converge to the global minimum of the problem. Then, the use of a global optimization algorithm is required. In this framework, stochastic or hybrid stochastic/deterministic methods are often used. Well known examples of the former are: pure random search, genetic algorithm, and simulated annealing. Among these methods,

the simplest approach is furnished by the pure random search, where a trial point is randomly generated at each iteration. It is accepted or rejected according to its performance: accepted if better than the current one, rejected otherwise. This simple procedure leads to a very high computational cost and several classes of global optimization algorithms have been developed in order to increase the efficiency of the search. One of them are the hybrid stochastic/deterministic methods where a local optimizer, such as the deterministic methods cited above, is combined with a global optimizer. For instance, when working with regular continuous objective functions, local optimizers can be turned into asymptotically global ones by restarting the search from a random initial point (Ritto et al. (2011)). Here, the probabilistic restart approach, where the restart procedure uses an adaptive probability density function constructed using the memory of past local searches, is employed (Luersen and Riche (2004)). The resulting optimization algorithm (coupling of the aforementioned restart and different local optimization algorithms) has been successfully applied to solve structural optimization problems (Luersen et al. (2004), Ritto et al. (2011), Torii et al. (2011)) and it is described in the sequel.

The local search is performed by a sequential quadratic programming (SQP) based algorithm available in the optimization toolbox of MATLAB. In this search, a starting point  $\underline{s}_0$  is chosen, then, a local search is performed and when a local minimum is found, the search is restarted. This restart procedure is described below.

The probability of having sampled a point  $\underline{s}$  is described by a Gaussian-Parzen-window approach Duda et al. (2001):

$$f(\underline{s}) = \frac{1}{M} \sum_{i=1}^M f_i(\underline{s}), \quad (4)$$

where  $M$  is the number of points  $\underline{s}_{(i)}$  already sampled. Such points come from the memory kept from the previous local searches, being, in the present version of the algorithm, all the starting points and local optima already found.  $f_i(\underline{s})$  is the Normal multivariate probability density function given by:

$$f_i(\underline{s}) = \frac{1}{(2\pi)^{n_p/2} \det([\Sigma])^{1/2}} \times \exp\left(-\frac{1}{2}(\underline{s} - \underline{s}_{(i)})^T [\Sigma]^{-1} (\underline{s} - \underline{s}_{(i)})\right), \quad (5)$$

where  $n_p$  is the problem dimension and  $[\Sigma]$  is the covariance matrix:

$$[\Sigma] = \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \sigma_{n_p}^2 \end{bmatrix}. \quad (6)$$

The variances are estimated by the relation:

$$\sigma_j^2 = \beta_o (s_j^{\max} - s_j^{\min})^2 \quad (7)$$

where  $\beta_o$  is a positive parameter that controls the length of the Gaussians, and  $s_j^{\max}$  and  $s_j^{\min}$  are the bounds of the  $j^{\text{th}}$  variable ( $j = 1, 2, \dots, n_p$ ). To keep the method simple, such variances are kept constant during the optimization process. At the end of each local search,  $N$  points are randomly sampled ( $\underline{s}_1, \underline{s}_2, \dots, \underline{s}_N$ ) and the one that minimizes Eq.(4) is selected as the initial point to restart the next local search. The stopping criterion of the global optimization of each subproblem is the maximum number of restarts ( $nr_{max}$ ) defined *a priori* by the user.

### 3.3 Summary of the optimization algorithm

Note that the global optimization strategy proposed here solves  $nr_{max}$  local search problems for different initial solutions. Since the local searches are performed using gradient based techniques, local optimality of the solutions can be ensured by applying strict tolerances during the optimization process. In other words, the use of gradient based techniques allows the algorithm to find local solutions that will likely be very close to some local optimum, since gradient based techniques present good performance for finding local optima. On the other hand, the probabilistic restart explores different areas of the domain searching for the global optimum. These are important advantages over other global optimization methods, such as some heuristics like genetic algorithms, since the latter need, in general, much higher computational effort in order to ensure local optimality when continuum design variables are used. In this context, it should be emphasized that the proposed approach present different benefits from most other global optimization strategies, since it does ensures local optimality to some degree. Besides it also yields a list of candidate local optima, which contains, with an increasing probability as the number of restarts increases, the global solution.

In order to make the implementation of the proposed algorithm easier, we describe it below in a stepwise manner.

- step 1. set  $nr_{max}$ ,  $\beta_o$ ,  $N$  and  $k = 1$ ;
- step 2. 1<sup>st</sup> local search: choose the starting point  $\underline{s}_{0(k)}$  randomly on the domain;
- step 3. save  $\underline{s}_{0(k)}$  and call the local optimizer, which returns the local optimum  $\underline{s}_{(k)}^*$ .
- step 4. save the local optimum  $\underline{s}_{(k)}^*$  and evaluate Eq.(4) for  $N$  randomly chosen points on the domain. The point which minimizes this equation is chosen as the next starting point, i.e.  $\underline{s}_{0(k+1)}$ .
- step 5. make  $k = k + 1$ ; if  $k > nr_{max}$  end the search; otherwise, return to step 3.

The reader should keep in mind that it is necessary to run the algorithm presented in this section for each NBI subproblem, i.e. it is run  $n_{sub}$  times to construct the PF.

## 4 NUMERICAL RESULTS

In this section, the global optimization algorithm proposed in Section 3 is employed and its results are analyzed and compared to the NSGA-II approach.

### 4.1 Mathematical Example

In this example, the following objective functions are to be minimized:

$$J(\underline{s}_1, \underline{s}_2) = \left\{ \begin{array}{l} 1 + (\phi_1(1, 2) - \phi_1(\underline{s}_1, \underline{s}_2))^2 + (\phi_2(1, 2) - \phi_2(\underline{s}_1, \underline{s}_2))^2 \\ (\underline{s}_1 + 3)^2 + (\underline{s}_2 + 1)^2 \end{array} \right\}$$

in which,

$$\phi_1(\underline{s}_1, \underline{s}_2) = 1/2\sin(\underline{s}_1) - 2\cos(\underline{s}_1) + \sin(\underline{s}_2) - 3/2\cos(\underline{s}_2)$$

and

$$\phi_2(\underline{s}_1, \underline{s}_2) = 3/2\sin(\underline{s}_1) - \cos(\underline{s}_1) + 2\sin(\underline{s}_2) - 1/2\cos(\underline{s}_2).$$

This problem has box constraints:  $-\pi \preceq \underline{s} \preceq \pi$ . An approximation of the surfaces generated by functions  $F_1$  and  $F_2$  are illustrated in Figures 2 and 3. From these Figures, one may easily see that the function  $J_1$  is not convex. Hence, this multi-objective optimization problem is suited for the proposed algorithm: NBI+restart. In order to evaluate the efficiency of the proposed algorithm, we compare its results to the NSGA-II. In this analysis, we fixed the total number of objective function evaluations (OFE) in 3000 in order to compare the results of both approaches. Also, we evaluate the gradients of the objective function numerically to make a fair comparison to the GA based method. The parameters employed in the NSGA-II algorithm were population and generation equal to 30 and 100, respectively. The parameters of the proposed algorithm are  $n_{sub} = 11$ ,  $nr_{max} = 5$ ,  $\beta_o = 0.10$  and  $N = 10$ .

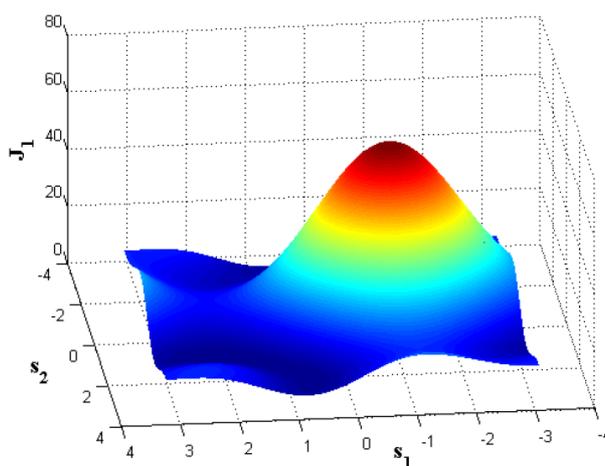


Figure 2: Approximation of the surface which represents  $J_1$ .

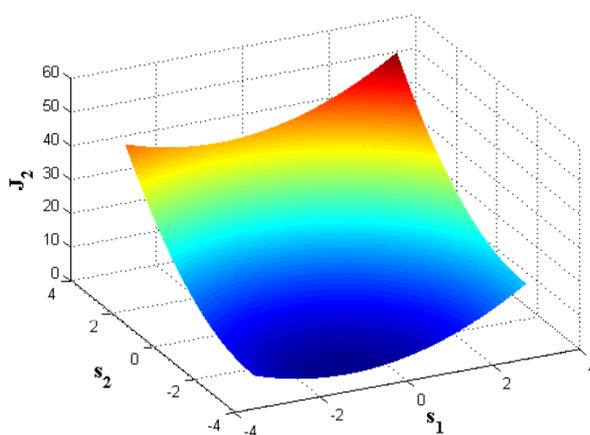


Figure 3: Approximation of the surface which represents  $J_2$ .

Figure 4 presents the PF obtained using the proposed algorithm and the NSGA-II. We can see from this figure that the proposed method is able to approximate the Pareto front of this multi-objective problem, generating an even spread of Pareto optimal points. Also, the PF of the NBI+restart approach was more even and presented slightly better results than the NSGA-II method. As commented at the end of Section 3.1, a point generated by the NBI may not be a

Pareto point if the boundary of the attained set in the objective space  $\mathcal{J}$  containing the Pareto points is non-convex, which is the case of this example. In Figure 4, we circled in blue the NBI points which are not Pareto points.

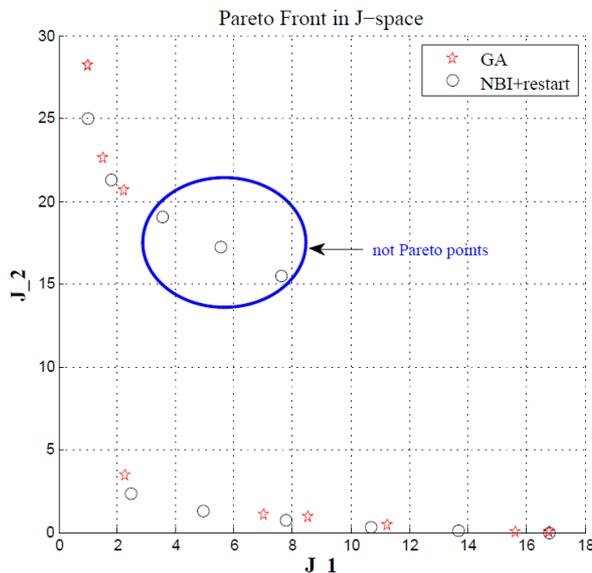


Figure 4: Pareto front obtained using the NBI+restart and the NSGA-II algorithms.

In this example, we showed that the proposed approach is able to handle the nonconvexity of the PF and the one of the objective functions. Also, its performance was slightly better than the well known NSGA-II algorithm.

#### 4.2 Engineering example: robust optimization of a rotor bearing system

In this section, we perform the robust optimization of a rotor bearing system. The rotor-bearing system analyzed is the one sketched in Fig. 5. Our goal here is to keep the natural frequencies of the rotor system as far as possible from the operational rotational speeds considering the uncertainties of system. We employ the penalty function presented by Ritto et al. (2011) for this purpose. It measures how close the natural frequencies of the system are from the rotational speeds that the rotor operates. That is, the higher its value, the closer they are. Thus, we aim at minimizing  $P_t$ . This penalty function  $P_t$  was shown to be nonconvex by their authors. The interested reader is referred to Ritto et al. (2011) for a complete description of how to evaluate  $P_t$ . The uncertainties of the rotor bearing system are taken into account in the optimization problem minimizing the mean and the variance of a penalization function  $P_t$ , resulting in a MOP. The problem is stated as:

$$\underline{\mathbf{s}}^* = \arg \min_{\underline{\mathbf{s}} \in \mathcal{C}} \mathbf{J}^{\text{stoc}}(\underline{\mathbf{s}}) = (E\{P_t(\underline{\mathbf{s}})\}, \text{Var}\{P_t(\underline{\mathbf{s}})\}), \quad (8)$$

where  $E\{P_t\}$  and  $\text{Var}\{P_t\}$  are respectively the expected value and variance of  $P_t$ , and  $\mathcal{C}$  is the set of the admissible values for  $\underline{\mathbf{s}}$ .  $\underline{\mathbf{s}}$  is comprised by the mean values of the parameters chosen as design variables, i.e. the mean of the mass, diameter and the bearing stiffness of the rotor. The only constraint is the range of the parameters  $\underline{\mathbf{s}}_{\min} \preceq \underline{\mathbf{s}} \preceq \underline{\mathbf{s}}_{\max}$ . The data used in the simulations are the same as in Ritto et al. (2011).

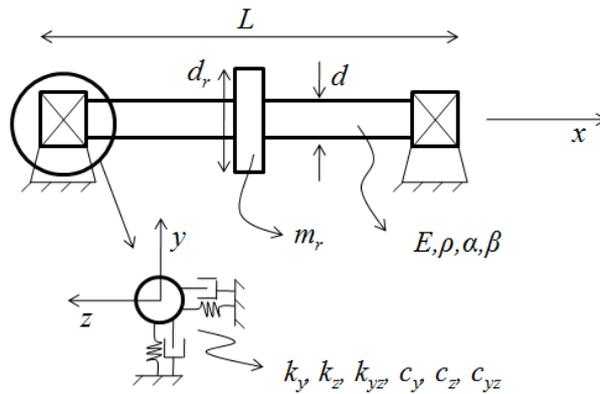


Figure 5: Rotor-bearing system.

The parameters used for the proposed algorithm are  $N=10$ ,  $\beta_o=0.01$ ,  $nr_{max}=8$  and  $n_{sub}=21$ . The parameters employed in the NSGA-II algorithm were population and generation equal to 50 and 1000, respectively. Monte Carlo simulation was employed in order to evaluate the mean  $E\{.}$  and variance  $Var\{.}$  operators of Eq.(8). In a previous work (Ritto et al. (2011)), it was shown that both operators converge around 500 simulations. Thus, it is necessary to call 500 times the finite element code for each evaluation of the MOP objective function.

The resulting PF using the proposed approach and the NSGA-II are shown in Fig. 6. One may easily see that the NSGA-II provided a slightly poorer approximation of the PF when compared to the NBI approach.

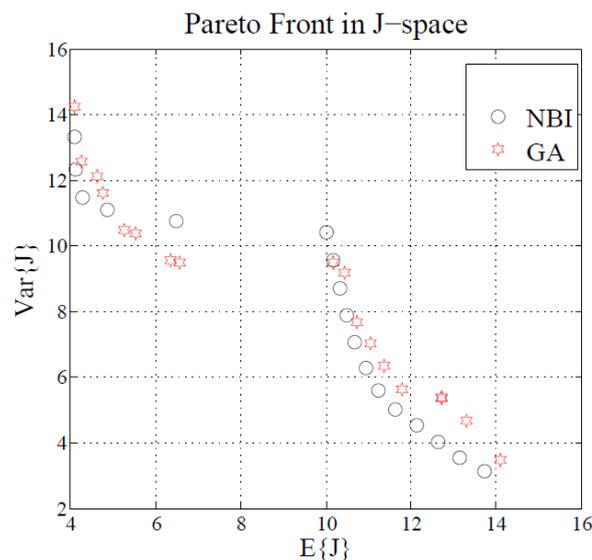


Figure 6: Pareto front of the robust optimization of the rotor-bearing system using the proposed method and NSGA-II.

Regarding the computational cost of the global optimization scheme, the number of calls of the finite element code necessary to obtain the 21 NBI points was  $10.6e6$ . To solve the same problem using the NSGA-II approach, it was necessary  $25.00e6$  calls of the finite element code. The NSGA-II required a computational cost 150% higher than the proposed approach. Thus,

the NBI + restart algorithm presented in this paper reached a better approximation of the PF requiring a much lower computational cost than the NSGA-II.

In this section, the advantages of the proposed approach over the NSGA-II in this engineering problem were discussed. The complexity of the MOP under analysis due to the shape of its PF, the nonconvexity of the objective functions and the computational cost involved was highlighted.

## 5 CONCLUSIONS

A new algorithm for the solution of MOP whose PF and the functions to be minimized are nonconvex was proposed in this paper. It applied the NBI to discretize the PF and it employed a global optimization algorithm based on a probabilistic restart procedure and local searches to minimize each NBI subproblem. The accuracy and efficiency of the proposed algorithm was shown in the numerical section comparing its results to the heuristic algorithms NSGA-II. It was shown that the proposed algorithm is able to handle MOP in which the PF and the functions to be minimized are not convex. Also, its accuracy and computational cost outperformed the ones of the NSGA-II algorithm.

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