Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXII, págs. 681-703 (artículo completo)
Carlos G. García Garino, Aníbal E. Mirasso, Mario A. Storti, Miguel E. Tornello (Eds.)
Mendoza, Argentina, 19-22 Noviembre 2013

SUBSPACE INVERSE POWER METHOD AND POLYNOMIAL CHAOS STRATEGY FOR THE MODAL RESPONSE OF RANDOM MECHANICAL SYSTEMS

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Keywords: Dynamic of structure, Frequency Response Function, Modal analysis, Random vibration, Propagation of uncertainty, Polynomial chaos

Abstract. This study concerns the computation of eigenelements of random matrices and dynamic frequency responses of linear stochastic mechanical systems. Two new strategies, based on transposing standard deterministic deflated inverse power method and subspace inverse power method into stochastic framework, are introduced via polynomial chaos expansion. Null and repeated-eigenvalue situations are addressed. Effectiveness of the proposed schemes is demonstrated through three simple examples.

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1 INTRODUCTION

This study concerns the propagation of uncertainties for structural dynamic problems. More specifically, stochastic solutions in the frequency domain are sought for linear problems when random properties are considered in the mechanical model in the inertial and elastic properties. Evaluating these random results are generally referred to as Stochastic Finite Element Methods (SFEM) (Ghanem and Spanos, 1991). For solving these stochastic problems, Monte Carlo Simulation (MCS) and spectral strategies enable to handle any kind of variations, of any size. The Polynomial Chaos (PC) expansion is a spectral representation of random processes which can be used in stochastic equations to represent unknown stochastic quantities. Since it uses a Hilbertian framework, the PC expansion is able to represent any second-order random quantity with any probability law.

Linear dynamic response is *e.g.* more efficiently dealt with in the modal space. To deal with the modal space one has first to solve an eigenvalue problem. This problem can either be a differential eigenvalue problem or a matrix eigenvalue problem, depending on whether a continuous model or a discrete model is used to describe the given system. For stochastic problems, probabilistic characterization of the eigensolutions of random matrix and of differential operators turn out to be of interest. SFEM with PC expansion are well developed for linear algebraic systems, but are less developed for the random eigenvalue and for the random response of dynamic problems. Challenging conditions for an eigenanalysis arise when part of the spectrum is clustered, that is, when there are repeated or closely spaced eigenvalues. This situation occurs frequently for complex mechanisms. In (Ghanem and Ghosh, 2007), the eigenvalue problem with distinct roots is rewritten as a set of non-linear equations and the methodology is latter extended in (Ghosh and Ghanem, 2008, 2012) to address the case of repeated roots. However, while the linear stochastic dynamic problem is carried out without approximation in this way, the computational effort involved appears to be more time consuming than the one involved in linear statics. Hence, to be more efficient numerically, the reference (Verhoosel et al., 2006) proposes a stochastic inverse power method for evaluating the random eigenvector associated to the lowest random eigenvalue. In (Verhoosel et al., 2006), a spectral shift strategy is carried out and normalization of normal modes is achieved only in the mean sense.

In this paper, discrete linear systems, or discretized continuous systems, having -or not- null or repeated eigenvalues are considered. The deterministic modal method, the deflated inverse power method, and the subspace inverse power method are transposed to the stochastic framework in an efficient way, from a numerical point of view. The organization of this paper is as follows. The first section addresses the modal dynamic problem, considering the deterministic and the stochastic frameworks. The second section recalls basic considerations of the PC representation and numerical strategies associated to them in the context of the dynamic responses, either from the direct or from the modal way. The third section addresses the proposed numerical strategies for the stochastic deflated inverse power method and the stochastic subspace inverse power method. Finally, three examples are chosen to demonstrate the effectiveness of the proposed methodologies.

2 PROBLEM STATEMENT

2.1 Deterministic problem

Let us considers the discrete –or discretized– linear deterministic problem of computing the the dynamic response of a system subjected to a forcing which is defined in the frequency

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domain ($\omega = 2\pi f$):

$$\left(\mathbf{k} - \omega^2\mathbf{m} + \mathrm{j}\mathbf{c}\omega\right)\mathbf{u}\left(\omega\right) = \mathbf{f}\left(\omega\right) \tag{1}$$

where $\mathbf{k}$, $\mathbf{m}$, and $\mathbf{c}$ are n-dimensional real symmetric structural matrices, being respectively the stiffness, mass, and damping matrices. The frequency response functions are $\mathbf{u}\left(\omega\right)$ when $\mathbf{f}\left(\omega\right)$ is chosen as the $n \times n$ identity matrix $\mathbf{i}$ over a specified frequency band, $\omega_{\min} \leq \omega \leq \omega_{\max}$.

Since the problem is linear with respect to the frequency response, it is interesting to introduce the modal representation associated to the above spatial representation. Restricting ourselves to proportional-damped systems, the modal representation is real and is defined as:

$$\begin{cases} \mathbf{k}\,\boldsymbol{\phi} = \mathbf{m}\,\boldsymbol{\phi}\,\boldsymbol{\lambda} \\ \boldsymbol{\phi}^{\mathrm{T}}\mathbf{m}\,\boldsymbol{\phi} = \boldsymbol{\mu} \end{cases} \tag{2}$$

with $\boldsymbol{\xi} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{c}\,\boldsymbol{\phi}$ and where:

- $\boldsymbol{\phi}$ is the normal-mode matrix formed with the eigenvectors as columns;
- $\boldsymbol{\lambda}$ is the diagonal matrix formed with the corresponding eigenvalues, that are the squares of the natural circular frequencies; and
- $\boldsymbol{\mu}$ is the diagonal modal mass matrix that is imposed to be the identity matrix, *i.e.* $\boldsymbol{\mu} = \mathbf{i}$, in this study.

Notice that a widely used convention consists in ordering the eigenvalues in ascending values, that is $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, while the associated eigenvectors are: $\boldsymbol{\phi} = \{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_n\}$. Hence, when there is clustered eigenvalues, switching of the ordering of eigenvectors can occur for a small change in the physical parameters. For example, considering distinct natural frequencies of a simple plate with varying dimensions, ordering of a torsional and a flexural mode can be switched due to the modification of these frequencies. However, these two modes can be recognized experimentally, when compared before and after modifications, because they keep their shapes. Thus, a Modal Assurance Criterion (MAC) will indicate that they are nearly parallel, showing then that they correspond to the same physical modes[1].

Then, the solution $\mathbf{u}\left(\omega\right)$ is such that:

$$\mathbf{u}\left(\omega\right) = \boldsymbol{\phi}\,\mathbf{x}\left(\omega\right) \tag{3}$$

for $\mathbf{x}\left(\omega\right)$ which satisfies the uncoupled modal equations:

$$\left(\boldsymbol{\lambda} - \omega^2\mathbf{i} + \mathrm{j}\boldsymbol{\xi}\omega\right)\mathbf{x}\left(\omega\right) = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{f}\left(\omega\right) \tag{4}$$

2.2 Stochastic problem

When the system has random parameters, its response becomes a stochastic process, $\mathbf{U}\left(\omega, \varpi\right)$, such that

$$\left(\mathbf{K}\left(\varpi\right) - \omega^2\mathbf{M}\left(\varpi\right) + \mathrm{j}\mathbf{C}\left(\varpi\right)\omega\right)\mathbf{U}\left(\omega, \varpi\right) = \mathbf{f}\left(\omega\right) \qquad \forall \varpi \in \Omega \tag{5}$$

[1]However, care must be taken when natural frequencies are repeated. In this situation any combination of associated normal modes leads to another also valid normal mode; that is the associated eigenspace has dimension bigger than one. In such case, shapes of both modes of the previous example looks like a mix of the two original modes. Then, a specific procedure must be involved to avoid difficulties in following their evolution.

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where $(\Omega, \mathcal{A}, \mathrm{Prob})$ is the probability space associated with the underlying physical experiments and where upper case letters denotes random variables corresponding to deterministic ones which are denote by the same letter in lower case. The stochastic eigenproblem associated to this model is:

$$\begin{cases} \mathbf{K}\,\mathbf{\Phi} = \mathbf{M}\,\mathbf{\Phi}\,\mathbf{\Lambda} \\ \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\,\mathbf{\Phi} = \boldsymbol{\mu} \end{cases} \tag{6}$$

where the argument ϖ is dropped for brevity. Then, $\mathbf{U}\,(\omega)$ can be express as:

$$\mathbf{U}\,(\omega) = \mathbf{\Phi}\,\mathbf{X}\,(\omega) \tag{7}$$

for:

$$\left(\mathbf{\Lambda} - \omega^2\mathbf{i} + \mathrm{j}\mathbf{\Xi}\omega\right)\mathbf{X}\,(\omega) = \mathbf{\Phi}^{\mathrm{T}}\mathbf{f}\,(\omega) \tag{8}$$

where:

$$\mathbf{\Lambda} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\,\mathbf{\Phi} \qquad \text{and} \qquad \mathbf{\Xi} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\,\mathbf{\Phi}. \tag{9}$$

One can notice that, due to the possible switching, or even mixing, of eigenvectors in the deterministic problem, the stochastic eigenproblem can be difficult to solve when there is clustered eigenvalues. Indeed, studying distributions of normal modes according to their ordering can have no physical meaning in this situation. Thus, to study the distribution of a physical mode, numerical tools to follow the evolution of the mode and the rotation of subspaces when there is eigenvalues with a multiplicity greater than one must be required. In this work, for the MCS, the MAC tool is chosen to avoid potential discontinuities in the stochastic eigenproblem.

3 PROPAGATION OF UNCERTAINTIES USING POLYNOMIAL CHAOS REPRESENTATION

Several methods can be applied to get a representation of $\mathbf{U}(\omega)$ for given distributions of the random input properties. From them, as an attractive alternative to the MCS, the PC representation consists in the expansion of the random process over a set of orthogonal polynomials. Wiener first introduced it in 1938 (Wiener, 1938) and used only Hermite polynomial basis to represent Gaussian random processes. More recently, works on generalized Polynomial Chaos propose a list of polynomial basis corresponding to non-Gaussian distributions (Xiu and Karniadakis, 2002). In effect, one can use any distribution as kernel and orthogonalize a polynomial basis of L_2 with respect to the probability measure generated by the chosen distribution. Ernst et al. (Ernst et al., 2012) demonstrated the L_2-convergence for any L_2 functionals, that is for any second-order random process (*i.e.* with finite variance).

3.1 Direct dynamic frequency response

Considering a generic random vector or random matrix, $\mathbf{G}$, the PC representation proposes to express it as a polynomial (truncation of a series) using a set of $n_g + 1$ orthogonal polynomials, denoted ψ_g , in variables ζ_i, $i \in \{1, \, , 2, \, \ldots \, d\}$, that are collected in a d-dimensional vector $\boldsymbol{\zeta}$:

$$\mathbf{G}(\varpi) = \sum_{g=0}^{n_g} \Psi_g(\boldsymbol{\zeta}(\varpi))\mathbf{g}_g \tag{10}$$

where $n_g + 1 = \frac{(d+h)!}{d!\,h!}$, h being the order of the expansion. Generally, for a practical implementation, the order of expansion results from a truncation which has to be chosen accordingly a

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suitable criterion. For the structural matrices $\mathbf{K}(\boldsymbol{\zeta})$, $\mathbf{M}(\boldsymbol{\zeta})$, or $\mathbf{C}(\boldsymbol{\zeta})$ and for responses $\mathbf{U}(\omega, \boldsymbol{\zeta})$ the adopted PC representation are:

$$\mathbf{K}(\boldsymbol{\zeta}) = \sum_{k=0}^{n_k} \Psi_k(\boldsymbol{\zeta})\,\mathbf{k}_k, \quad \mathbf{M}(\boldsymbol{\zeta}) = \sum_{m=0}^{n_m} \Psi_m(\boldsymbol{\zeta})\,\mathbf{m}_m, \quad \mathbf{C}(\boldsymbol{\zeta}) = \sum_{c=0}^{n_c} \Psi_c(\boldsymbol{\zeta})\,\mathbf{c}_c \tag{11}$$

and:

$$\mathbf{U}(\omega, \boldsymbol{\zeta}) = \sum_{u=0}^{n_u} \Psi_u \mathbf{u}_u(\omega)\,. \tag{12}$$

Coming back to the generic random vector or matrix $\mathbf{G}$, the n_g vectors or matrices of deterministic coefficients $\mathbf{g}_g$, now used to describe $\mathbf{G}$, can be evaluated in two ways: using an intrusive or a non-intrusive method. The non-intrusive method can be always used to find the coefficients of the stochastic variables when they are represented by a PC expansion. It uses the orthogonality of the polynomials with respect to the appropriated inner product to evaluate each vector of deterministic coefficients $\mathbf{g}_g$:

$$\mathbf{g}_g = \frac{<\mathbf{G}(\boldsymbol{\zeta}), \Psi_g(\boldsymbol{\zeta})>}{<\Psi_g(\boldsymbol{\zeta}), \Psi_g(\boldsymbol{\zeta})>} \tag{13}$$

where $<\mathbf{G}(\boldsymbol{\zeta}), \Psi_g(\boldsymbol{\zeta})> = \int_{\mathbb{R}^d} \mathbf{G}(\boldsymbol{\zeta})\,\Psi_g(\boldsymbol{\zeta})\,p_{\boldsymbol{\zeta}}(\boldsymbol{\zeta})\,\mathrm{d}\boldsymbol{\zeta}$ represents the inner product introduced by the PDF $p_{\boldsymbol{\zeta}}(\boldsymbol{\zeta})$. To evaluate this inner product, it is introduced a partition, $\boldsymbol{\zeta}_i$, the integrations points, $\mathbf{w}$ the vector which collects the quadrature weights and n_{gp} the number of points, the above integration can be done using a Gauss quadrature rule:

$$<\mathbf{G}(\boldsymbol{\zeta}), \Psi_g(\boldsymbol{\zeta})> = \sum_{i=1}^{n_{\mathrm{GP}}} \mathbf{G}(\boldsymbol{\zeta}_i)\,\Psi_g(\boldsymbol{\zeta}_i)\,\{\mathbf{w}\}_i\,. \tag{14}$$

In this way, the residual associated to the difference between $\mathbf{G}(\varpi)$ and its PC representation is orthogonal to its PC representation. This strategy to determine the representation can be referred to as a strong characterization (Ghosh and Ghanem, 2012) and can be used to find coefficients of input stochastic variables such as $\mathbf{k}_k$, $\mathbf{m}_m$ or $\mathbf{c}_c$ for the structural matrices as well as to model the output stochastic variables such as $\mathbf{u}_u(\omega)$ for responses $\mathbf{U}(\omega, \boldsymbol{\zeta})$.

The intrusive method, or weak characterization, follows a variational approach. As a first step, the PC representation introduced for the stochastic variables are put into the governing equations. For instance, relation 5 produces:

$$\left(\sum_{k=0}^{n_k} \Psi_k \mathbf{k}_k - \omega^2 \sum_{m=0}^{n_m} \Psi_m \mathbf{m}_m + \mathrm{j} \sum_{c=0}^{n_c} \Psi_c \mathbf{c}_c \omega\right) \left(\sum_{u=0}^{n_u} \Psi_u \mathbf{u}_u(\omega)\right) = \mathbf{f}(\omega) \tag{15}$$

Next, the vectors coefficients are given by solving the system which is obtained by taking expectations of the equations produced by the projection of relation 15 onto the set of polynomials Ψ:

$$\sum_{u=0}^{n_u} \left(\sum_{k=0}^{n_k} \langle \Psi_i \Psi_k \Psi_u \rangle \mathbf{k}_k - \omega^2 \sum_{m=0}^{n_m} \langle \Psi_i \Psi_m \Psi_u \rangle \mathbf{m}_m + \mathrm{j} \sum_{c=0}^{n_c} \langle \Psi_i \Psi_c \Psi_u \rangle \mathbf{c}_c \omega\right) \mathbf{u}_u(\omega) = \langle \Psi_i \rangle \mathbf{f}(\omega) \qquad i = 0, \ldots, n_u \tag{16}$$

This leads to a deterministic matricial system of the form:

$$\left(\mathcal{K} - \omega^2 \mathcal{M} + \mathrm{j}\mathcal{C}\omega\right) \mathcal{U}(\omega) = \mathcal{F}(\omega) \tag{17}$$

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where:

$$\mathcal{K} = \begin{bmatrix} \sum_{k=0}^{n_k} \langle \Psi_0 \Psi_k \Psi_0 \rangle \, \mathbf{k}_k & \sum_{k=0}^{n_k} \langle \Psi_0 \Psi_k \Psi_1 \rangle \, \mathbf{k}_k & \cdots \\ \sum_{k=0}^{n_k} \langle \Psi_1 \Psi_k \Psi_0 \rangle \, \mathbf{k}_k & \sum_{k=0}^{n_k} \langle \Psi_1 \Psi_k \Psi_1 \rangle \, \mathbf{k}_k & \\ \vdots & & \ddots \end{bmatrix} \qquad \mathcal{U} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \end{bmatrix} \qquad \mathcal{F} = \begin{bmatrix} \langle \Psi_0 \rangle \, \mathbf{f} \\ \langle \Psi_1 \rangle \, \mathbf{f} \\ \vdots \end{bmatrix} \tag{18}$$

The matrices $\mathcal{M}$ and C have similar form as the matrix $\mathcal{K}$. Relation 17 is a linear system of deterministic equations which involves expectations of three PC products.

From a numerical point of view, the weak characterization provides a set of $n \times n_g$ coupled algebraic equations, while the strong characterization determines $\mathbf{g}_g$ from integrations, requiring at least a set of experiments for a quadrature rule. When the order of expansion is adequate, the coefficients obtained from both characterizations should be identical when the same orthogonal basis is considered. However, this is not true for an inadequate truncation since the weak characterization couples the coefficients, while the strong characterization determines each coefficient independently.

3.2 Modal frequency response

The direct strategy proposed in the previous section by using the intrusive method is expensive numerically when considering a large frequency band since it involves solving a large system many times. As the structural matrices do not depend on the frequency, the modal response presented in subsection 2.2 is relevant. Considering only a particular, single mode, $\mathbf{\Phi}_r$, modal quantities can be represented in the chosen basis as:

$$\mathbf{\Phi}_r(\boldsymbol{\zeta}) = \sum_{\phi=0}^{n_\lambda} \Psi_\phi(\boldsymbol{\zeta}) \, \boldsymbol{\phi}_{r\phi}, \quad \text{and} \quad \mathbf{\Lambda}_r(\boldsymbol{\zeta}) = \sum_{\lambda=0}^{n_\lambda} \Psi_\lambda(\boldsymbol{\zeta}) \, \boldsymbol{\lambda}_{r\lambda} \tag{19}$$

and the relation 6 , taken in the weak sense, becomes:

$$\begin{cases} \sum_{\phi=0}^{n_\lambda} \sum_{k=0}^{n_k} \langle \Psi_i \Psi_k \Psi_\phi \rangle \, \mathbf{k}_k \, \boldsymbol{\phi}_{r\phi} = \sum_{\phi=0}^{n_\lambda} \sum_{\lambda=0}^{n_\lambda} \sum_{m=0}^{n_m} \langle \Psi_i \Psi_m \Psi_\phi \Psi_\lambda \rangle \, \mathbf{m}_m \, \boldsymbol{\phi}_{r\phi} \, \boldsymbol{\lambda}_{r\lambda} \\ \sum_{\phi=0}^{n_\lambda} \sum_{m=0}^{n_m} \sum_{\lambda=0}^{n_\lambda} \langle \Psi_i \Psi_\phi \Psi_m \Psi_\lambda \rangle \, \boldsymbol{\phi}_{r\phi}^{\mathrm{T}} \mathbf{m}_m \boldsymbol{\phi}_{r\lambda} = \langle \Psi_i \rangle \, \boldsymbol{\mu} \end{cases} \quad i = 0, \ldots, n_\lambda \tag{20}$$

which can be rewritten in a matricial form as:

$$\begin{cases} \mathcal{K} \, \Phi_r = \mathcal{M}' \, \Phi_r \, \Lambda_r \\ \Phi_r^{\mathrm{T}} \mathcal{M}' \Phi_r = \Upsilon \end{cases} \tag{21}$$

when collecting:

$$\Phi_r = \begin{bmatrix} \boldsymbol{\phi}_{r0} \\ \boldsymbol{\phi}_{r1} \\ \vdots \end{bmatrix} \qquad \Lambda_r = \begin{bmatrix} \boldsymbol{\lambda}_{r0} \\ \boldsymbol{\lambda}_{r1} \\ \vdots \end{bmatrix} \qquad \Upsilon = \begin{bmatrix} \langle \Psi_0 \rangle \, \boldsymbol{\mu} \\ \langle \Psi_1 \rangle \, \boldsymbol{\mu} \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle \Psi_0 \rangle \, \mathbf{i} \\ \mathbf{0} \\ \vdots \end{bmatrix} \tag{22}$$

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and:

$$\mathcal{M}'\Phi_r = \begin{bmatrix} \sum_{\phi=0}^{n_\lambda}\sum_{m=0}^{n_m} \langle \Psi_0\Psi_k\Psi_\phi\Psi_0\rangle \mathbf{m}_m\boldsymbol{\phi}_{r\phi} & \sum_{\phi=0}^{n_\lambda}\sum_{m=0}^{n_m} \langle \Psi_0\Psi_k\Psi_\phi\Psi_1\rangle \mathbf{m}_m\boldsymbol{\phi}_{r\phi} & \cdots \\ \sum_{\phi=0}^{n_\lambda}\sum_{m=0}^{n_m} \langle \Psi_1\Psi_k\Psi_\phi\Psi_0\rangle \mathbf{m}_m\boldsymbol{\phi}_{r\phi} & \sum_{\phi=0}^{n_\lambda}\sum_{m=0}^{n_m} \langle \Psi_1\Psi_k\Psi_\phi\Psi_1\rangle \mathbf{m}_m\boldsymbol{\phi}_{r\phi} & \\ \vdots & & \ddots \end{bmatrix}. \tag{23}$$

This is a non-linear system of deterministic equations which involves expectations of four PC products. The reference (Ghanem and Ghosh, 2007) proposes to solve it using a Newton-Raphson method starting from a statistically sampled initial value.

Then, synthesis of $\mathbf{U}(\omega)$ leads to consider:

$$\mathbf{U}(\omega) = \sum_{r=1}^{n}\sum_{\phi=0}^{n_\lambda}\Psi_\phi\boldsymbol{\phi}_{r\phi}\mathbf{X}_r(\omega) \tag{24}$$

for $\mathbf{X}_r(\omega) = \sum_{x=0}^{n_x}\Psi_x\,\mathbf{x}_{rx}(\omega)$ such that:

$$\left(\boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi}_r - \omega^2\boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi}_r + \mathrm{j}\boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{C}_r\boldsymbol{\Phi}_r\omega\right)\mathbf{X}_r(\omega) = \boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{f}(\omega) \tag{25}$$

or:

$$\left(\boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi}_r - \omega^2 + \mathrm{j}\boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi}_r\omega\right)\mathbf{X}_r(\omega) = \boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{f}(\omega) \tag{26}$$

since normal modes are mass normalized. It is, in the weak sense:

$$\sum_{x=0}^{n_x}\left(\sum_{\phi=0}^{n_\lambda}\sum_{k=0}^{n_k}\sum_{\lambda=0}^{n_\lambda}\langle\Psi_i\Psi_\phi\Psi_k\Psi_\lambda\Psi_x\rangle\boldsymbol{\phi}_{r\phi}^{\mathrm{T}}(\mathbf{k}_k + \mathrm{j}\omega\mathbf{c}_k)\boldsymbol{\phi}_{r\lambda} - \omega^2\langle\Psi_i\Psi_x\rangle\right)\mathbf{x}_{rx}(\omega) = \left\langle\Psi_i^2\right\rangle\boldsymbol{\phi}_{ri}^{\mathrm{T}}\mathbf{f}(\omega) \qquad i = 0,\ldots,n_x \tag{27}$$

This is a linear system of deterministic equations which involves expectations of five PC products. Then it is more efficient computationally to use PC representation of modal quantities. Using $\boldsymbol{\Lambda}_r = \sum_{\lambda=0}^{n_\lambda}\Psi_\lambda\boldsymbol{\lambda}_{r\lambda}$ and $\boldsymbol{\Xi}_r = \sum_{\xi=0}^{n_\xi}\Psi_\xi\boldsymbol{\xi}_{r\xi}$, the above expression is rewritten:

$$\left(\boldsymbol{\Lambda}_r - \omega^2 + \mathrm{j}\boldsymbol{\Xi}_r\omega\right)\mathbf{X}_r(\omega) = \boldsymbol{\Phi}_r^{\mathrm{T}}\mathbf{f}(\omega) \tag{28}$$

to produce, in the weak sense:

$$\sum_{x=0}^{n_x}\left(\sum_{\lambda=0}^{n_\lambda}\langle\Psi_i\Psi_\lambda\Psi_x\rangle\boldsymbol{\lambda}_{r\lambda} - \omega^2\langle\Psi_i\Psi_x\rangle + \mathrm{j}\omega\sum_{\xi=0}^{n_\xi}\langle\Psi_i\Psi_\xi\Psi_x\rangle\boldsymbol{\xi}_{r\xi}\right)\mathbf{x}_{rx}(\omega) = \left\langle\Psi_i^2\right\rangle\boldsymbol{\phi}_{ri}^{\mathrm{T}}\mathbf{f}(\omega) \qquad i = 0,\ldots,n_x \tag{29}$$

which is a linear system of deterministic equations which involves expectations of three PC products. This leads to a deterministic matricial system of the form:

$$\left(\mathcal{J}_r - \omega^2\mathcal{I}_r + \mathrm{j}\mathcal{D}_r\omega\right)\mathcal{X}_r(\omega) = \mathcal{R}_r(\omega) \tag{30}$$

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where:

$$\mathcal{J}_r = \begin{bmatrix} \sum_{\lambda=0}^{n_\lambda} \langle \Psi_0 \Psi_\lambda \Psi_0 \rangle \boldsymbol{\lambda}_{r\lambda} & \sum_{\lambda=0}^{n_\lambda} \langle \Psi_0 \Psi_\lambda \Psi_1 \rangle \boldsymbol{\lambda}_{r\lambda} & \cdots \\ \sum_{\lambda=0}^{n_\lambda} \langle \Psi_1 \Psi_\lambda \Psi_0 \rangle \boldsymbol{\lambda}_{r\lambda} & \sum_{\lambda=0}^{n_\lambda} \langle \Psi_1 \Psi_\lambda \Psi_1 \rangle \boldsymbol{\lambda}_{r\lambda} & \\ \vdots & & \ddots \end{bmatrix} \quad \mathcal{X}_r = \begin{bmatrix} x_{r0} \\ x_{r1} \\ \vdots \end{bmatrix} \quad \mathcal{R}_r = \begin{bmatrix} \langle \Psi_0^2 \rangle \boldsymbol{\phi}_{r0}^{\mathrm{T}} \mathbf{f} \\ \langle \Psi_1^2 \rangle \boldsymbol{\phi}_{r1}^{\mathrm{T}} \mathbf{f} \\ \vdots \end{bmatrix} \tag{31}$$

while matrices $\mathcal{I}_r$ and D_r have a similar form to the matrix $\mathcal{J}_r$. Thus, to evaluate $\mathbf{U}(\omega)$ once the stochastic eigenproblem is solved, assuming it provides the PC representation of eigenvectors and eigenvalues, it is sufficient to solve 27 to obtain the modal displacement matrix coefficients (that is a vector, if $\mathbf{f}(\omega)$ also is). It corresponds to the equation of a single degree of freedom system, thanks to the normal modes basis that decouples mechanical degrees of freedoms.

4 NUMERICAL METHODS FOR THE DETERMINATION OF NORMAL MODES BASIS

Considering the deterministic normal modes basis problem, it exists numerical methods dedicated to its efficient determination ((Géradin and Rixen, 1997)).

4.1 Deterministic methods

From a numerical point of view, Lanczos method and power method are adapted to find eigenvectors of large numerical problems. The inverse power method is the most basic method of computing an eigenvector $\boldsymbol{\phi}_1$ associated to the lowest eigenvalue $|\boldsymbol{\lambda}_1| > 0$ for the generalized problem 2 when considering a positive-definite stiffness matrix. The algorithm of this method is:

1. Choose an initial vector $\boldsymbol{\phi}_1^{(0)}$ such that $\boldsymbol{\phi}_1^{(0)\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}_1^{(0)} = 1$,
2. Compute $\boldsymbol{\lambda}_1^{(0)} = \boldsymbol{\phi}_1^{(0)\mathrm{T}} \mathbf{k}\, \boldsymbol{\phi}_1^{(0)}$
3. For $k = 1, 2, \ldots$ do:
   (a) Let $\mathbf{f}_1^{(k)} = \mathbf{m}\, \boldsymbol{\phi}_1^{(k-1)}$
   (b) Solve $\mathbf{k}\, \boldsymbol{\phi}_1^{(k)} = \mathbf{f}_1^{(k)}$
   (c) Normalize $\boldsymbol{\phi}_1^{(k)}$ such that $\left\| \boldsymbol{\phi}_1^{(k)} \right\|_{\mathbf{m}} = \boldsymbol{\phi}_1^{(k)\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}_1^{(k)} = 1$
   (d) Compute $\boldsymbol{\lambda}_1^{(k)} = \boldsymbol{\phi}_1^{(k)\mathrm{T}} \mathbf{k}\, \boldsymbol{\phi}_1^{(k)}$
4. End

This algorithm repeats the step 3 until $\boldsymbol{\lambda}_1^{(k)}$ converges to within some tolerance ϵ, being $\left| \lambda_r^{(k)} - \lambda_r^{(k-1)} \right| < \epsilon \left| \lambda_r^{(k)} \right|$. If it converges, it converges to the eigenvector $\boldsymbol{\phi}_1$ corresponding to the lowest eigenvalue $|\boldsymbol{\lambda}_1|$. Noticed that even if $\boldsymbol{\phi}_1^{(0)}$ is orthogonal to $\boldsymbol{\phi}_1$, it is expected that $\boldsymbol{\phi}_1$ would be recovered numerically due to round-off errors. However, the power method fails to converge if $|\lambda_p| = |\lambda_1|$, but $\lambda_p \neq \lambda_1$. In addition, convergence is not ensured when there are repeated or null eigenfrequencies: the convergence depends on the starting guess vector.

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Then, deflation can be carried out to construct the next lowest eigenvalue $|\boldsymbol{\lambda}_2|$ and so on for the next ones. In such a way, the eigenvalue $|\boldsymbol{\lambda}_p|$, for $p > 1$, is obtained by introducing an orthogonalization step before the normalization, as:

$$\boldsymbol{\phi}_p^{(k)} = \boldsymbol{\phi}_p^{(k)} - \sum_{q=1}^{p-1} \boldsymbol{\phi}_q \boldsymbol{\phi}_q^{\mathrm{T}} \mathbf{m} \boldsymbol{\phi}_p^{(k)} \tag{32}$$

This orthogonalization step is also necessary when considering semi-definite stiffness matrix where the first eigenvectors correspond to the null space of $\mathbf{k}$, in which case a generalized inverse is applied within iterations. The deflated inverse power method algorithm corresponding to this situation is:

1. Find the null space $\boldsymbol{\phi}^0$ of $\mathbf{k}$ such that $\boldsymbol{\phi}^{0\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}^0 = \mathbf{i}$ and let $\boldsymbol{\phi}_q \leftarrow \boldsymbol{\phi}^0$
2. For each sought mode r, do:
   (a) Choose an initial vector $\boldsymbol{\phi}_r^{(0)}$ such that $\boldsymbol{\phi}_q^{\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}_r^{(0)} = \mathbf{0}$ and $\boldsymbol{\phi}_r^{(0)\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}_r^{(0)} = 1$,
   (b) Compute $\boldsymbol{\lambda}_r^{(0)} = \boldsymbol{\phi}_r^{(0)\mathrm{T}} \mathbf{k}\, \boldsymbol{\phi}_r^{(0)}$
   (c) For $k = 1, 2, \ldots$ do:
      i. Let $\mathbf{f}_r^{(k)} = \mathbf{m}\, \boldsymbol{\phi}_r^{(k-1)}$
      ii. Solve $\mathbf{k}\, \boldsymbol{\phi}_r^{(k)} = \mathbf{f}_r^{(k)}$
      iii. Orthonormalize $\boldsymbol{\phi}_r^{(k)}$, that is such that $\boldsymbol{\phi}_q^{\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}_r^{(k)} = \mathbf{0}$ and $\left\| \boldsymbol{\phi}_r^{(k)} \right\|_{\mathbf{m}} = \boldsymbol{\phi}_r^{(k)\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}_r^{(k)} = 1$
      iv. Compute $\boldsymbol{\lambda}_r^{(k)} = \boldsymbol{\phi}_r^{(k)\mathrm{T}} \mathbf{k}\, \boldsymbol{\phi}_r^{(k)}$
   (d) Let $\boldsymbol{\phi}_q \leftarrow \left[\boldsymbol{\phi}_q,\, \boldsymbol{\phi}_r^{(k)} \right]$
3. End

However, this algorithm does not ensure the convergence of the eigenvectors when there are repeated eigenvalues. In practice, it depends on the choice made for the starting guess vector. The subspace inverse power method is an alternative to this algorithm to effectively deal with the repeated eigenvalues case, independently of the choice of the starting guess vector. This method consists in handling simultaneously a set of n_r modes $\boldsymbol{\phi}^{(k)}$, while preserving their orthonormality within iterations. The algorithm is:

1. Find the null space $\boldsymbol{\phi}^0$ of $\mathbf{k}$ such that $\boldsymbol{\phi}^{0\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}^0 = \mathbf{i}$
2. Choose an orthonormalized initial matrix $\boldsymbol{\phi}^{(0)}$ such that $\boldsymbol{\phi}^{0\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}^{(0)} = \mathbf{0}$ and $\boldsymbol{\phi}^{(0)\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}^{(0)} = \mathbf{i}$
3. Compute $\boldsymbol{\lambda}^{(0)} = \boldsymbol{\phi}^{(0)\mathrm{T}} \mathbf{k}\, \boldsymbol{\phi}^{(0)}$
4. For $k = 1, 2, \ldots$ until convergence of $\boldsymbol{\lambda}^{(k)}$ do:
   (a) Let $\mathbf{f}^{(k)} = \mathbf{m}\, \boldsymbol{\phi}^{(k-1)}$
   (b) Solve $\mathbf{k}\, \boldsymbol{\phi}^{(k)} = \mathbf{f}^{(k)}$
   (c) Orthonormalize $\boldsymbol{\phi}^{(k)}$ such that $\boldsymbol{\phi}^{0\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}^{(k)} = \mathbf{0}$ and $\boldsymbol{\phi}^{(k)\mathrm{T}} \mathbf{m}\, \boldsymbol{\phi}^{(k)} = \mathbf{i}$
   (d) Compute $\boldsymbol{\lambda}^{(k)} = \boldsymbol{\phi}^{(k)\mathrm{T}} \mathbf{k}\, \boldsymbol{\phi}^{(k)}$
5. End

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4.2 Determination of the stochastic normal modes basis using the deflated inverse power method and polynomial chaos representation

The deflated inverse power method is a suitable alternative to the Newton-Raphson method to solve the system 20 in order to build the stochastic normal mode basis.

The proposed algorithm for the deflated inverse power method is the following:

1. For each sought mode r, do:
   (a) Choose an initial vector $\Phi_r^{(0)}$ to represents $\mathbf{\Phi}_r^{(0)}$
   (b) Find $\Phi_r^{(0)1}$ with a unit modal mass from $\Phi_r^{(0)}$ and let $\Phi_r^{(0)} \leftarrow \Phi_r^{(0)1}$
   (c) Evaluate $\Lambda_r^{(0)}$ from a modal projection of $\mathbf{K}$ over $\mathbf{\Phi}_r^{(0)}$
   (d) If $\left|\Lambda_r^{(0)}\right| > 0$ do for $k = 1, 2, \ldots$ until convergence of $\Lambda_r^{(k)}$:
      i. Let $\mathcal{F}_r^{(k)} = \mathcal{M}\, \Phi_r^{(k-1)}$
      ii. Solve $\mathcal{K}\, \Phi_r^{(k)} = \mathcal{F}_r^{(k)}$
      iii. If $r > 1$, find $\Phi_r^{(k)\perp}$ that orthogonalizes $\mathbf{\Phi}_r^{(k)}$ to $\mathbf{\Phi}_q$ for $q \in \{1, \ldots, r-1\}$ and let $\Phi_r^{(k)} \leftarrow \Phi_r^{(k)\perp}$
      iv. Evaluate $\Phi_r^{(k)1}$ that normalizes modal mass to unity and let $\Phi_r^{(k)} \leftarrow \Phi_r^{(k)1}$
      v. Evaluate $\Lambda_r^{(k)}$ from a modal projection of $\mathbf{K}$ over $\Phi_r^{(k)}$
2. End

A simple starting guess for $\Phi_r^{(0)}$ can be produced from the normalized normal mode $\boldsymbol{\phi}_r$ of the deterministic nominal mechanical system, in which case $\mathbf{\Phi}_{r0}^{(0)} = \boldsymbol{\phi}_r$ while $\mathbf{\Phi}_{r\phi}^{(0)} = \mathbf{0}$ for $\phi > 0$. However, it has to be noted that even when the deterministic eigenvector is effectively mass normalized, its stochastic counterpart may not be. Then, the normalization substep (b) of this algorithm holds in any cases. Modal projections, orthogonalization, and normalization steps are detailed in the following subsections. Notice that choosing the suggested starting guess for $\Phi_r^{(0)}$ enables the algorithm to deal with null eigenvalues, in which case the substep (d) is bypassed.

4.3 Modal projections

Modal projections of structural matrices are performed easily in a strong characterization. Considering a mass-normalized normal mode $\mathbf{\Phi}_q$ and the stiffness matrix $\mathbf{K}$, the modal projection $\mathbf{\Lambda}_q$ is defined as:

$$\mathbf{\Lambda}_q(\boldsymbol{\zeta}) = \mathbf{\Phi}_q(\boldsymbol{\zeta})\, \mathbf{K}(\boldsymbol{\zeta})\, \mathbf{\Phi}_q(\boldsymbol{\zeta}) \tag{33}$$

while its PC representation is: $\mathbf{\Lambda}_q(\boldsymbol{\zeta}) = \sum_{\lambda=0}^{n_\lambda} \Psi_\lambda(\boldsymbol{\zeta})\, \boldsymbol{\lambda}_{q\lambda}$ for: $\boldsymbol{\lambda}_{q\lambda} = \dfrac{< \mathbf{\Lambda}_q(\boldsymbol{\zeta}), \Psi_\phi(\boldsymbol{\zeta}) >}{< \Psi_\phi^2(\boldsymbol{\zeta}) >}$.

They are given numerically from a quadrature rule:

$$\boldsymbol{\lambda}_{q\lambda} = \frac{1}{< \Psi_\lambda^2(\boldsymbol{\zeta}) >} \sum_{i=1}^{n_{\mathrm{gp}}} \mathbf{\Lambda}_q(\boldsymbol{\zeta}_i)\, \Psi_\phi(\boldsymbol{\zeta}_i)\, \mathbf{w}_i \tag{34}$$

where:

$$\mathbf{\Lambda}_q(\boldsymbol{\zeta}_i) = \sum_{\phi=0}^{n_\lambda} \sum_{k=0}^{n_k} \sum_{\lambda=0}^{n_\lambda} \Psi_\phi(\boldsymbol{\zeta_i})\, \Psi_m(\boldsymbol{\zeta}_i)\, \Psi_\lambda(\boldsymbol{\zeta}_i)\, \boldsymbol{\phi}_{q\phi}^{\mathrm{T}} \mathbf{k}_k \boldsymbol{\phi}_{q\lambda}. \tag{35}$$

An identical strategy enables the computation of modal damping.

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4.4 Orthogonalization step

The orthogonalization step consists in searching the PC representation of $\mathbf{\Phi}_r^{(k)\perp}$ orthogonal to $\mathbf{\Phi}_q$, when knowing the PC representation of $\mathbf{\Phi}_q$ and the one of $\mathbf{\Phi}_r^{(k)}$ for $q < r$. For mass-normalized normal modes $\mathbf{\Phi}_q$, the orthogonalization condition is defined by:

$$\mathbf{\Phi}_q^{\mathrm{T}}\mathbf{M}\,\mathbf{\Phi}_r^{(k)\perp} = 0 \tag{36}$$

and is achieved for:

$$\mathbf{\Phi}_r^{(k)\perp} = \mathbf{\Phi}_r^{(k)} - \mathbf{\Phi}_q\mathbf{\Phi}_q^{\mathrm{T}}\mathbf{M}\,\mathbf{\Phi}_r^{(k)} \tag{37}$$

which is for the PC representation of the stochastic variables:

$$\sum_{\phi=0}^{n_\lambda}\Psi_\phi\left(\boldsymbol{\zeta}\right)\boldsymbol{\phi}_{r\phi}^{(k)\perp} = \sum_{\phi=0}^{n_\lambda}\Psi_\phi\left(\boldsymbol{\zeta}\right)\boldsymbol{\phi}_{r\phi}^{(k)} - \sum_{\phi=0}^{n_\lambda}\sum_{\varphi=0}^{n_\lambda}\sum_{m=0}^{n_m}\sum_{\lambda=0}^{n_\lambda}\Psi_\phi\left(\boldsymbol{\zeta}\right)\Psi_\varphi\left(\boldsymbol{\zeta}\right)\Psi_m\left(\boldsymbol{\zeta}\right)\Psi_\lambda\left(\boldsymbol{\zeta}\right)\boldsymbol{\phi}_{q\phi}\boldsymbol{\phi}_{q\varphi}^{\mathrm{T}}\mathbf{m}_m\boldsymbol{\phi}_{r\lambda}^{(k)} \tag{38}$$

Then, $\boldsymbol{\phi}_{r\phi}^{(k)\perp}$ can be found straightforwardly from:

$$\sum_{\phi=0}^{n_\lambda}\left\langle\Psi_i\Psi_\phi\right\rangle\boldsymbol{\phi}_{r\phi}^{(k)\perp} = \sum_{\phi=0}^{n_\lambda}\left\langle\Psi_i\Psi_\phi\right\rangle\boldsymbol{\phi}_{r\phi}^{(k)} - \sum_{\phi=0}^{n_\lambda}\sum_{\varphi=0}^{n_\lambda}\sum_{m=0}^{n_m}\sum_{\lambda=0}^{n_\lambda}\left\langle\Psi_i\Psi_\phi\Psi_\varphi\Psi_m\Psi_\lambda\right\rangle\boldsymbol{\phi}_{q\phi}\boldsymbol{\phi}_{q\varphi}^{\mathrm{T}}\mathbf{m}_m\boldsymbol{\phi}_{r\lambda}^{(k)} \qquad i = 0,\ldots,n_\lambda \tag{39}$$

that is:

$$\boldsymbol{\phi}_{ri}^{(k)\perp} = \boldsymbol{\phi}_{ri}^{(k)} - \frac{1}{\left\langle\Psi_i^2\right\rangle}\sum_{\varphi=0}^{n_\lambda}\sum_{\phi=0}^{n_\lambda}\sum_{m=0}^{n_m}\sum_{\lambda=0}^{n_\lambda}\left\langle\Psi_i\Psi_\phi\Psi_\varphi\Psi_m\Psi_\lambda\right\rangle\boldsymbol{\phi}_{q\phi}\boldsymbol{\phi}_{q\varphi}^{\mathrm{T}}\mathbf{m}_m\boldsymbol{\phi}_{r\lambda}^{(k)} \qquad i = 0,\ldots,n_\lambda \tag{40}$$

However, the above expression involve expectations of five PC products. Hence, an enhanced –more efficient– strategy is desirable from a numerical point of view to find $\boldsymbol{\phi}_{r\phi}^{(k)\perp}$. From:

$$\sum_{\phi=0}^{n_\lambda}\Psi_\phi\left(\boldsymbol{\zeta}_i\right)\boldsymbol{\phi}_{r\phi}^{(k)\perp} = \mathbf{\Phi}_r^{(k)\perp}\left(\boldsymbol{\zeta}_i\right) \qquad \forall i \tag{41}$$

we know that:

$$\boldsymbol{\phi}_{r\phi}^{(k)\perp} = \frac{<\mathbf{\Phi}_r^{(k)\perp},\Psi_\phi\left(\boldsymbol{\zeta}\right)>}{<\Psi_\phi^2\left(\boldsymbol{\zeta}\right)>} \tag{42}$$

or:

$$\boldsymbol{\phi}_{r\phi}^{(k)\perp} = \frac{1}{<\Psi_\phi^2\left(\boldsymbol{\zeta}\right)>}\sum_{i=1}^{n_{\mathrm{gp}}}\left(\Psi_\phi\left(\boldsymbol{\zeta}_i\right)\mathbf{w}_i\right)\left(\Psi_\phi\left(\boldsymbol{\zeta}_i\right)\boldsymbol{\phi}_{r\phi}^{(k)\perp}\right) \tag{43}$$

when sampling points $\boldsymbol{\zeta}_i$ are the ones of the Gauss quadrature. To express $\left(\Psi_\phi\left(\boldsymbol{\zeta}_i\right)\boldsymbol{\phi}_{r\phi}^{(k)\perp}\right)$, it is known from the relation 37 that:

$$\mathbf{\Phi}_r^{(k)\perp}\left(\boldsymbol{\zeta}\right) = \sum_{\phi=0}^{n_\lambda}\Psi_\phi\left(\boldsymbol{\zeta}\right)\boldsymbol{\phi}_{r\phi}^{(k)} - \sum_{\phi=0}^{n_\lambda}\sum_{\varphi=0}^{n_\lambda}\sum_{m=0}^{n_m}\sum_{\lambda=0}^{n_\lambda}\Psi_\phi\left(\boldsymbol{\zeta}\right)\Psi_\varphi\left(\boldsymbol{\zeta}\right)\Psi_m\left(\boldsymbol{\zeta}\right)\Psi_\lambda\left(\boldsymbol{\zeta}\right)\boldsymbol{\phi}_{q\phi}\boldsymbol{\phi}_{q\varphi}^{\mathrm{T}}\mathbf{m}_m\boldsymbol{\phi}_{r\lambda}^{(k)} \tag{44}$$

and we state that the above equality holds if:

$$\left(\Psi_\phi\left(\boldsymbol{\zeta}_j\right)\boldsymbol{\phi}_{r\phi}^{(k)\perp}\right) = \Psi_\phi\left(\boldsymbol{\zeta}_j\right)\boldsymbol{\phi}_{r\phi}^{(k)} - \sum_{\varphi=0}^{n_\lambda}\sum_{m=0}^{n_m}\sum_{\lambda=0}^{n_\lambda}\Psi_\phi\left(\boldsymbol{\zeta}_j\right)\Psi_\varphi\left(\boldsymbol{\zeta}_j\right)\Psi_m\left(\boldsymbol{\zeta}_j\right)\Psi_\lambda\left(\boldsymbol{\zeta}_j\right)\boldsymbol{\phi}_{q\phi}\boldsymbol{\phi}_{q\varphi}^{\mathrm{T}}\mathbf{m}_m\boldsymbol{\phi}_{r\lambda}^{(k)} \qquad \forall\zeta_j \tag{45}$$

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Hence, evaluating the above relation at sampling points $\boldsymbol{\zeta}_i$ enables the evaluation of $\boldsymbol{\phi}_{r\phi}^{(k)\perp}$ from the relation 43.

For large structural systems, this strategy is suitable to construct orthogonalized statics or inertial modes that are necessary to enrich a truncated normal mode basis, as well as for modal synthesis methods (Géradin and Rixen, 1997).

4.5 Normalization step

The normalization step consists in searching $\boldsymbol{\Phi}_r^{(k)1}$ which is defined by:

$$\boldsymbol{\Phi}_r^{(k)1\mathrm{T}}\mathbf{M}\,\boldsymbol{\Phi}_r^{(k)1} = 1 \tag{46}$$

It is achieved for:

$$\boldsymbol{\Phi}_r^{(k)1} = \frac{1}{\left\|\boldsymbol{\Phi}_r^{(k)}\right\|_{\mathbf{M}}}\boldsymbol{\Phi}_r^{(k)} \tag{47}$$

where: $\left\|\boldsymbol{\Phi}_r^{(k)}\right\|_{\mathbf{M}} = \left(\boldsymbol{\Phi}_r^{(k)\mathrm{T}}\mathbf{M}\,\boldsymbol{\Phi}_r^{(k)}\right)^{1/2}$ while $\boldsymbol{\Phi}_r^{(k)1}(\boldsymbol{\zeta}) = \sum_{\phi=0}^{n_\lambda}\Psi_\phi(\boldsymbol{\zeta})\,\boldsymbol{\phi}_{r\phi}^{(k)1}$ with $\boldsymbol{\phi}_{r\phi}^{(k)1} = \dfrac{<\boldsymbol{\Phi}_r^{(k)\perp},\Psi_\phi(\boldsymbol{\zeta})>}{<\Psi_\phi^2(\boldsymbol{\zeta})>}$.

In practice, as for the enhanced formulation proposed for the orthogonalization step, it can be performed from a quadrature rule:

$$\boldsymbol{\phi}_{r\phi}^{(k)1} = \frac{1}{<\Psi_\phi^2(\boldsymbol{\zeta})>}\sum_{i=1}^{n_{\mathrm{gp}}}\boldsymbol{\Phi}_r^{(k)1}(\boldsymbol{\zeta}_i)\,\Psi_\phi(\boldsymbol{\zeta}_i)\,\mathbf{w}_i. \tag{48}$$

where:

$$\boldsymbol{\Phi}_r^{(k)1}(\boldsymbol{\zeta}_i) = \left(\boldsymbol{\Phi}_r^{(k)\mathrm{T}}(\boldsymbol{\zeta}_i)\,\mathbf{M}(\boldsymbol{\zeta}_i)\,\boldsymbol{\Phi}_r^{(k)}(\boldsymbol{\zeta}_i)\right)^{-1/2}\boldsymbol{\Phi}_r^{(k)}(\boldsymbol{\zeta}_i) \tag{49}$$

This is an enhanced formulation from a numerical point of view since:

- it does not involved the non-linear solution of the equations obtained by the weak characterization;
- it does not requires expectations of a four PC terms product as it is by the weak characterization.

4.6 Determination of the stochastic normal modes basis using the subspace inverse power method and polynomial chaos representation

The subspace inverse power method is a suitable alternative to the deflated inverse power method to solve the system 20 in order to build the stochastic normal modes basis.

The proposed algorithm is the following. For an initial set of n_r modes $\boldsymbol{\Phi}^{(0)}$ having the PC representation $\Phi^{(0)}$, do:

1. Orthonormalize initial matrix $\Phi^{(0)}$
2. Evaluate $\Lambda^{(0)}$ from a modal projection of $\mathbf{K}$ over $\Phi^{(0)}$ and let $\Lambda_{\mathbf{q}}^{(k)}$ and $\Phi_{\mathbf{q}}^{(0)}$ the subset of $\Lambda^{(k)}$ and $\Phi^{(0)}$ which corresponds to non null eigenvalues
3. For $k = 1, 2, \ldots$ until convergence of $\Lambda_{\mathbf{q}}^{(k)}$ do:

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(a) Let $\mathcal{F}_{\mathbf{q}}^{(k)} = \mathcal{M}\, \Phi_{\mathbf{q}}^{(k-1)}$

(b) Solve $\mathcal{K}\, \Phi_{\mathbf{q}}^{(k)} = \mathcal{F}_{\mathbf{q}}^{(k)}$

(c) Orthonormalize $\mathbf{\Phi}^{(k)}$

(d) Evaluate $\Lambda_{\mathbf{q}}^{(k)}$ from a modal projection of $\mathbf{K}$ over $\Phi_{\mathbf{q}}^{(k)}$

4. End

In this algorithm, the substep (c) implies the orthogonalization of all normal modes, including the rigid body modes, when they exist. The starting guess $\Phi^{(0)}$ is produced from the normal mode matrix ϕ of the deterministic nominal mechanical system. Orthonormalization is achieved as in the above subsections 4.4 and 4.5:

1. Evaluate $\Phi_1^{(k)1}$ that normalizes modal mass to unity and let $\Phi_1^{(k)} \leftarrow \Phi_1^{(k)1}$
2. For $r \leftarrow 2$ to n_r,

   (a) Find $\Phi_r^{(k)\perp}$ that orthogonalizes $\mathbf{\Phi}_r^{(k)}$ to $\mathbf{\Phi}_p^{(k)}$ for $p \in \{1, \ldots, r-1\}$ and let $\Phi_r^{(k)} \leftarrow \Phi_r^{(k)\perp}$

   (b) Find $\Phi_r^{(k)1}$ that normalizes the modal mass to unity and let $\Phi_r^{(k)} \leftarrow \Phi_r^{(k)1}$

3. End

and the convergence condition for $\Lambda_{\mathbf{q}}$ becomes: $\max\limits_{r\in\{1,\ldots,n_r\}} \left(\frac{\left\| \Lambda_r^{(k)} - \Lambda_r^{(k-1)} \right\|}{\left\| \Lambda_r^{(k)} \right\|} \right) < \epsilon$.

5 NUMERICAL APPLICATIONS

5.1 First application

The first application concerns a system having two degree of freedom and involving three random variables. Introducing ζ_l for $l = \{1, 2, 3\}$, three uncorrelated standard normal variables, the random structural matrices are:

$$\mathbf{K} = \mu_a \left(1 + \frac{\zeta_1^2 - 1}{\sqrt{2}} \frac{\sigma_a}{\mu_a} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{50}$$

and:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \frac{\sigma_a}{\mu_a} \begin{bmatrix} \zeta_2^2 - 1 & 0 \\ 0 & \zeta_3^2 - 1 \end{bmatrix}. \tag{51}$$

The parameters are: $\mu_a = 20$ and $\frac{\sigma_a}{\mu_a} = 5$ %. Notice that positive definiteness is ensured for the stiffness and masses as long as $\frac{\sigma_a}{\mu_a} < \sqrt{2}$.

Hermite polynomials up to the second degree (10 terms) are used for the PC representation of the stochastic eigensolutions. Eigensolutions of the nominal system are:

$$\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\lambda} = \begin{bmatrix} 0 & 0 \\ 0 & 40 \end{bmatrix} \tag{52}$$

showing that the first normal mode is a rigid body mode. Then, considering the stochastic eigenproblem, the first eigenvalue is deterministic, indeed it is null. The amplitude of the normalized rigid-body mode is however stochastic since it has a unit norm with respect to an inner product defined by the stochastic mass metric. The PC representation of this first normal mode

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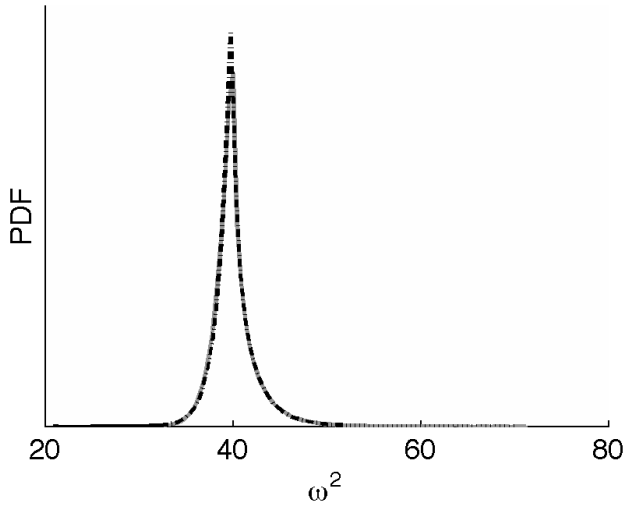

Figure 1: PDF of the second eigenvalue for the first application; thick grey lines are for MCS, dashed black lines are for PC expansion.

is obtained by normalizing the starting guest which is built from the nominal system. The second normal mode is sought from the deflated inverse power method presented above, using the enhanced strategy for the orthogonal step. Since the iteration matrix $\mathcal{K}$ is not invertible, a Moore-Penrose pseudo inverse is involved for the generation of a new iterate. When starting from the nominal second mode the convergence towards the stochastic second normal mode is reached in two iterations. Figure 1 shows the PDFs of the second eigenvalue and Figure 2 shows the PDFs of the eigenvectors components obtained. These results are compared to a MCS which has been carried out using 5×10^5 sample size. A satisfactory agreement is achieved.

5.2 Second application

The second application is inspired by the simplified three DOFs model of the bladed disk of the reference (Ghosh and Ghanem, 2012). Introducing ζ_l for $l = \{1,\ 2,\ 3,\ 4\}$, four uncorrelated standard normal variables, the random structural matrices are:

$$\mathbf{K} = \mu_b \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\sigma_b}{\sqrt{2}} \begin{bmatrix} \zeta_1^2 - 1 & 0 & 0 \\ 0 & \zeta_2^2 - 1 & 0 \\ 0 & 0 & \zeta_3^2 - 1 \end{bmatrix} + \mu_c \left(1 + \frac{\zeta_4^2 - 1}{\sqrt{2}} \frac{\sigma_c}{\mu_c}\right) \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{53}$$

and:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \frac{\sigma_b}{\mu_b} \begin{bmatrix} \zeta_1^2 - 1 & 0 & 0 \\ 0 & \zeta_2^2 - 1 & 0 \\ 0 & 0 & \zeta_3^2 - 1 \end{bmatrix} \tag{54}$$

The parameters used are the ones proposed in the reference (Ghosh and Ghanem, 2012): $\mu_b = 100$, $\mu_c = 20$, while $\frac{\sigma_b}{\mu_b} = \frac{\sigma_c}{\mu_c} = 5$ % and Hermite polynomials up to the fourth order (70 terms) are used for the PC representation of the stochastic eigensolutions and of the stochastic dynamic responses.

The deflated inverse power method presented above requires 30, 39, and 3 iterations to converge towards the three stochastic normal modes when the normal modes of the nominal system

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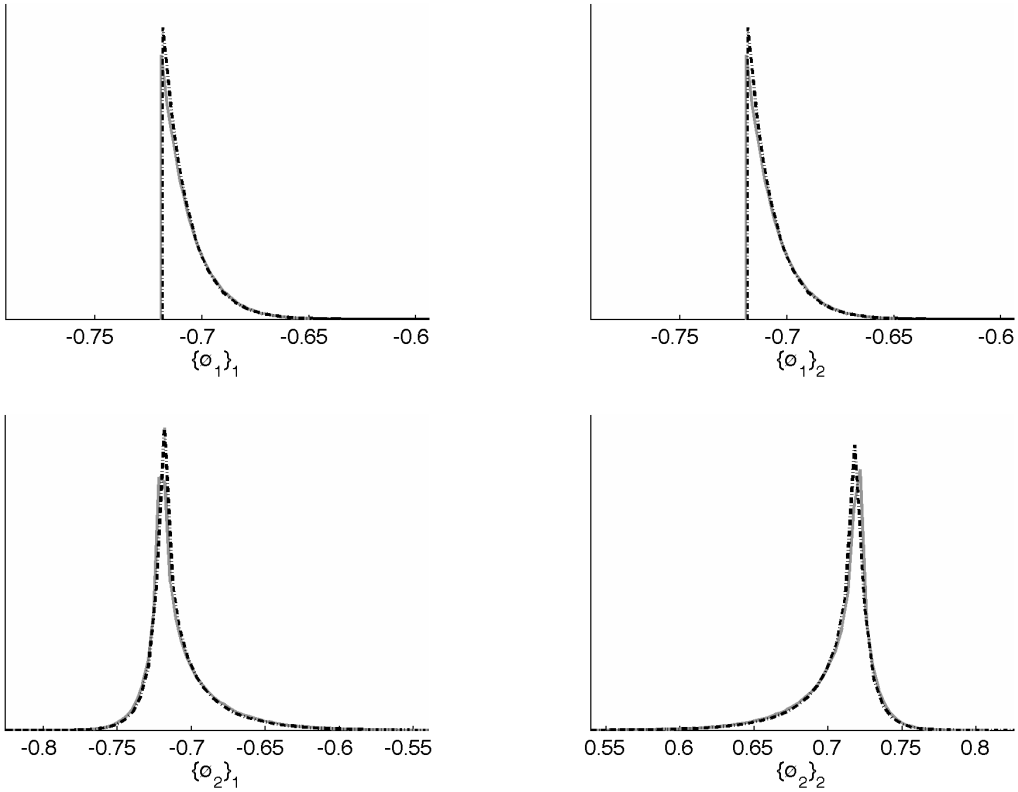

Figure 2: PDFs of the two components (column-wise) of the two normal modes (row-wise) for the first application; thick grey lines are for MCS, dashed black lines are for PC expansion.

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Normal mode r	1	2	3
Mean	111.7	140.0	168.5
Standard deviation	0.681	2.40	3.87

Table 1: Statistics of normal modes frequencies for the second application.

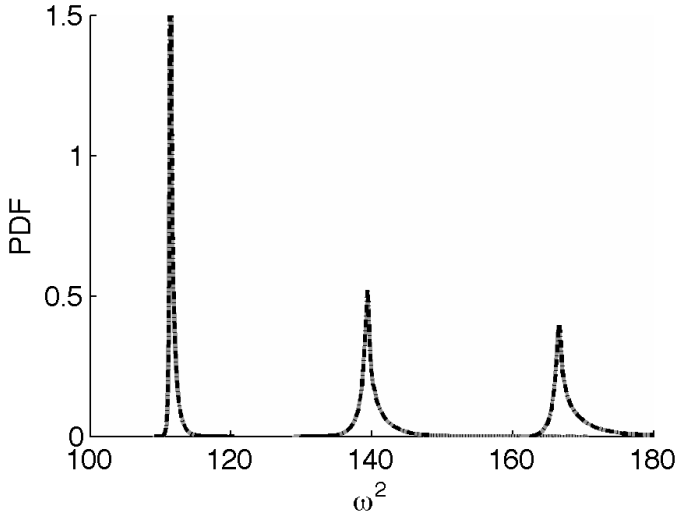

Figure 3: PDFs of the three eigenvalues for the second application; thick grey lines are for MCS, dashed black lines are for PC expansion.

are used as starting guesses. These normal modes are:

$$\boldsymbol{\phi} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \tag{55}$$

Results for means and standard deviations of stochastic eigenvalues are given in Table 1 and their PDFs are plotted in Figure 3. CDFs of the normal modes components are shown in Figure 4. The obtained results are compared to a MCS which has been carried out using 5×10^5 sample size and a Modal Assurance Criterion to identify and to order the stochastic normal modes.

To compute frequency responses functions, the structural forces and damping matrices are $\mathbf{f}(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Figure 5 and 6 show i) the mean; ii) the 5 percentile; and iii) the 95 percentile of the amplitude of the nine frequency responses functions, as well as their second statistical moment. Notice that computing the mean amplitude requires all the terms of the PC expansion, and not only the first one. Hence, agreement with MCS can only be achieved for an adequate truncation of the PC expansion. For the stochastic eigenproblem, a satisfactory agreement is achieved. However, discrepancies occur at some frequencies ranges for the standard deviation of the amplitude response. This indicates that the order of truncation is inadequate in these ranges for the frequency responses, while it is adequate for the stochastic normal modes.

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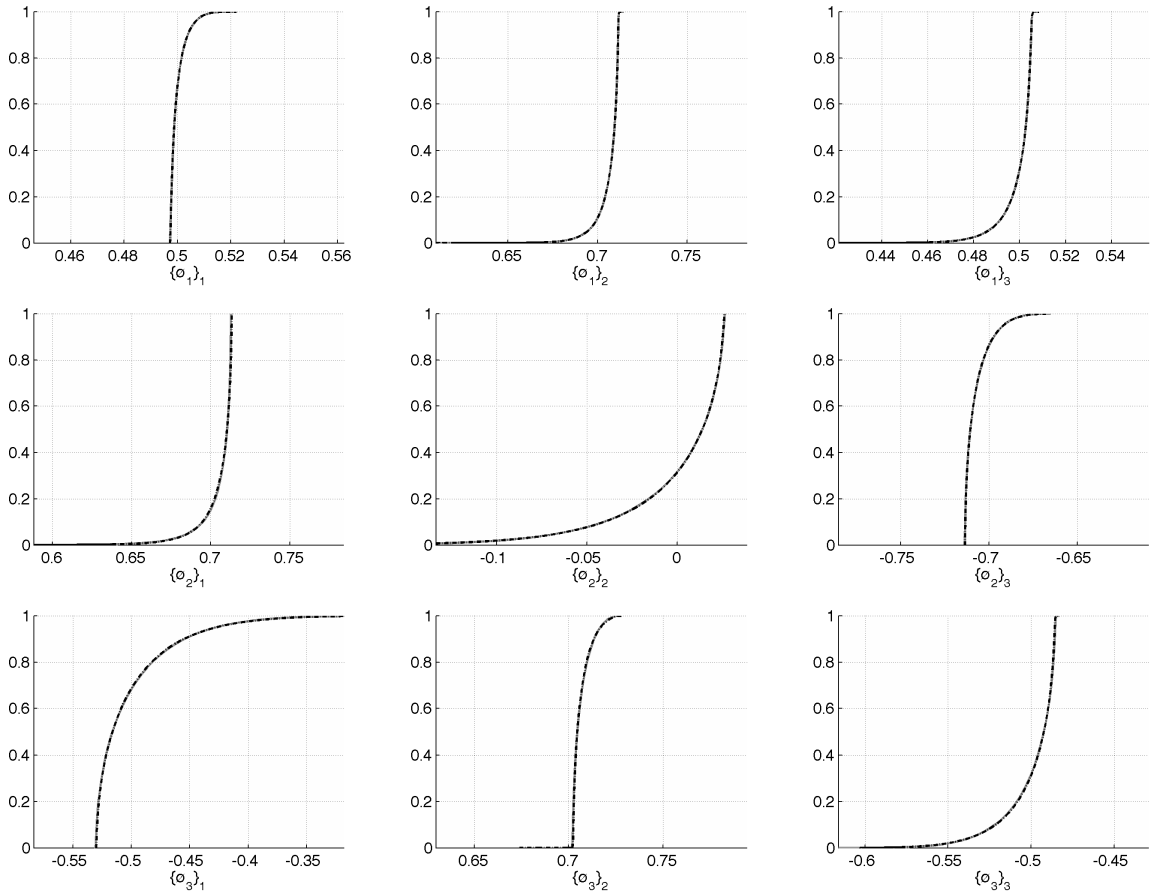

Figure 4: CDFs of the three components (column-wise) of the three normal modes (row-wise) for the second application; thick grey lines are for MCS, dashed black lines are for PC expansion.

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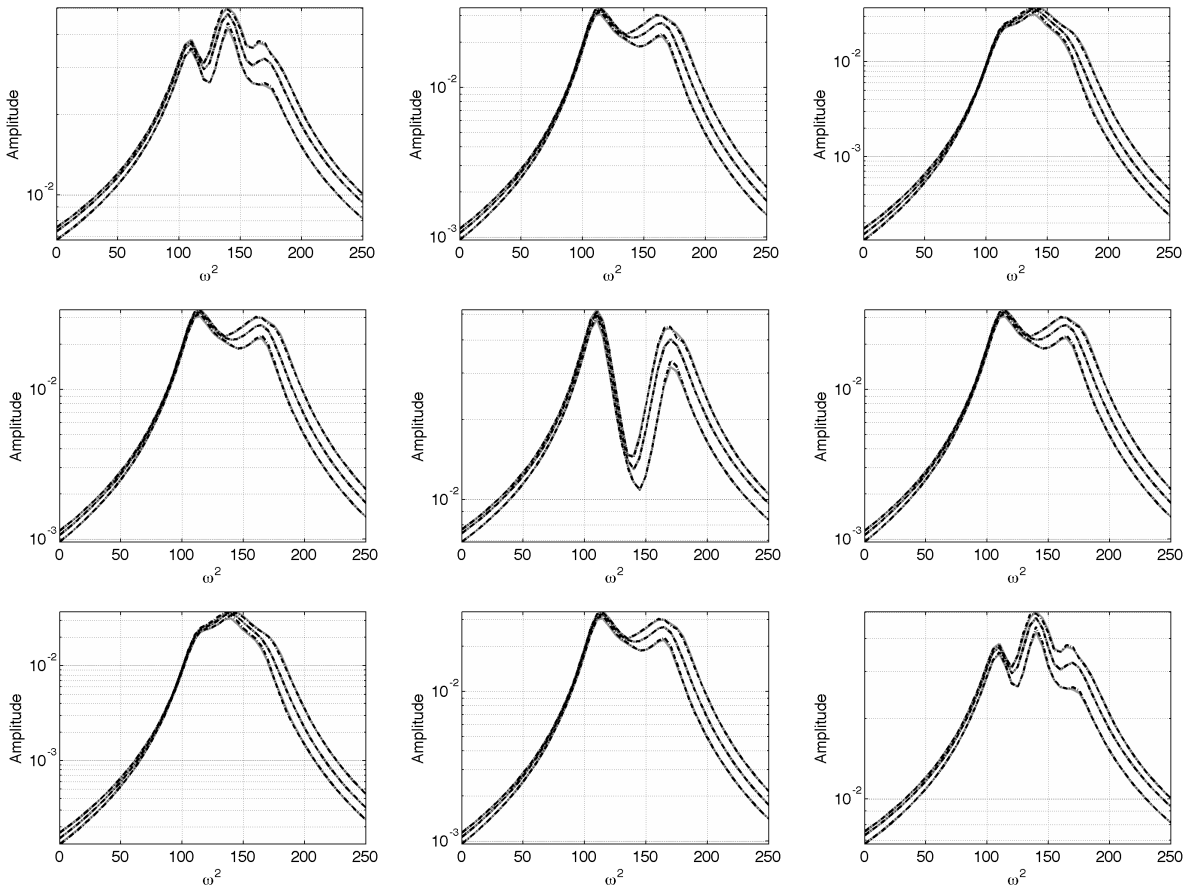

Figure 5: Frequency responses amplitudes of the three DoF system for the second application using the PC expansion up to the 4th order; Mean, 5% and 95 % lines of confidence interval are represented; From up to down is for the first to the last degree of freedom; From left-hand to right-hand is for an excitation applied from the first to the last degree of freedom; Thick grey lines are for MCS, dashed black lines are for PC expansion.

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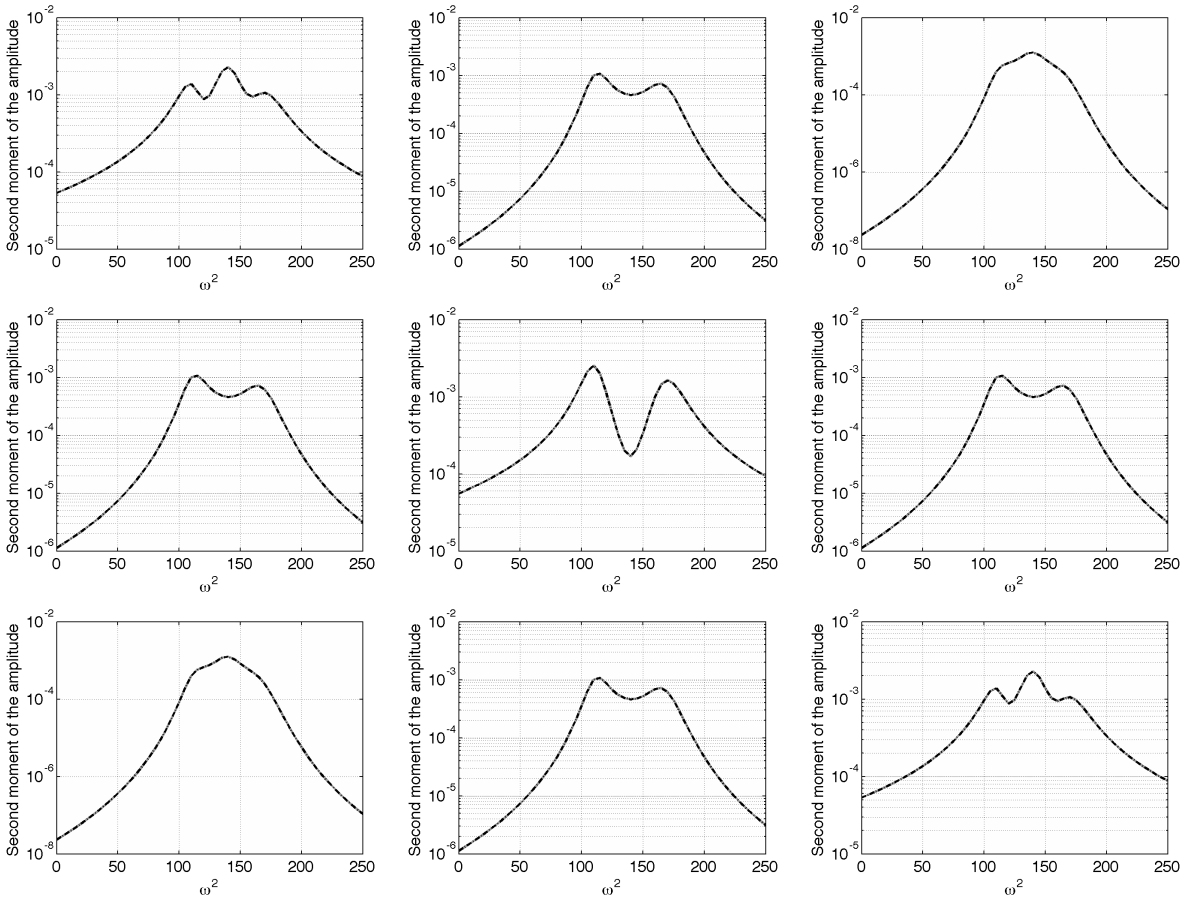

Figure 6: Second moment of frequency responses amplitudes of the three DoF system for the second application using the PC expansion up to the 4th order; From up to down is for the first to the last degree of freedom; From left-hand to right-hand is for an excitation applied from the first to the last degree of freedom; Thick grey lines are for MCS, dashed black lines are for PC expansion.

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Normal mode r	1	2	3
Mean	99.75	159.4	160.8
Standard deviation	2.51	4.55	5.54

Table 2: Statistics of normal modes frequencies for the third application.

5.3 Third application

The third application is the simplified three DOFs model of the bladed disk of the reference (Ghosh and Ghanem, 2012). Introducing ζ_l for $l = \{1, 2, 3, 4\}$, four uncorrelated standard normal variables, the random structural matrices are:

$$\mathbf{K} = \mu_b \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\sigma_b}{\sqrt{2}} \begin{bmatrix} \zeta_1^2 - 1 & 0 & 0 \\ 0 & \zeta_2^2 - 1 & 0 \\ 0 & 0 & \zeta_3^2 - 1 \end{bmatrix} + \mu_c \left(1 + \frac{\zeta_4^2 - 1}{\sqrt{2}} \frac{\sigma_c}{\mu_c}\right) \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{56}$$

and $\mathbf{M}$ is the identity. The parameters are: $\mu_b = 100$, $\mu_c = 20$, while $\frac{\sigma_b}{\mu_b} = \frac{\sigma_c}{\mu_c} = 5\ \%$ and Hermite polynomials up to the fourth degree are used for the PC representation of the stochastic eigensolutions and for the PC representation of the stochastic dynamic responses. Eigensolutions of the deterministic system are:

$$\boldsymbol{\phi} = \frac{1}{6} \begin{bmatrix} 2\sqrt{3} & \sqrt{6} & 3\sqrt{2} \\ 2\sqrt{3} & \sqrt{6} & -3\sqrt{2} \\ 2\sqrt{3} & -2\sqrt{6} & 0 \end{bmatrix} \qquad \boldsymbol{\lambda} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 160 & 0 \\ 0 & 0 & 160 \end{bmatrix} \tag{57}$$

This application is interesting since two of the three normal modes have the same natural frequency.

In this situation, the stochastic deflated inverse power method has difficulties to converge, hence the stochastic subspace inverse power method is preferred. Finding the three stochastic modes with this method requires seventy five iterations for a maximum relative precision $\epsilon = 10^{-6}$ for the eigenfrequencies when starting from eigenvectors of the nominal system. Results of means and standard deviations of stochastic eigenvalues obtained are given in Table 2.

To compute frequency responses functions, the structural forces, and the damping matrices are $\mathbf{f}(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Figures 7 and 8 show i) the mean; ii) the 5 percentile; and iii) the 95 percentile of the amplitudes of the nine frequency response functions, as well as their second statistical moments. The obtained results are compared to a MCS which has been carried out using 5×10^5 sample size. A satisfactory agreement is achieved, especially around the resonant frequencies, showing that eigenvalues multiplicity is handled effectively.

6 CONCLUSION

This study concerns stochastic discretized linear dynamic problems that may have large variations. It is limited to symmetrical structural matrices and proportional damping. The normal mode basis is chosen to express the structural spatial frequency responses and it is done in an efficient way. The PC representation is the chosen spectral approach for random quantities since it can effectively handle any stochastic variations. Within this framework, from the normal basis construction up to the modal synthesis, we propose efficient strategies in order to maintain the

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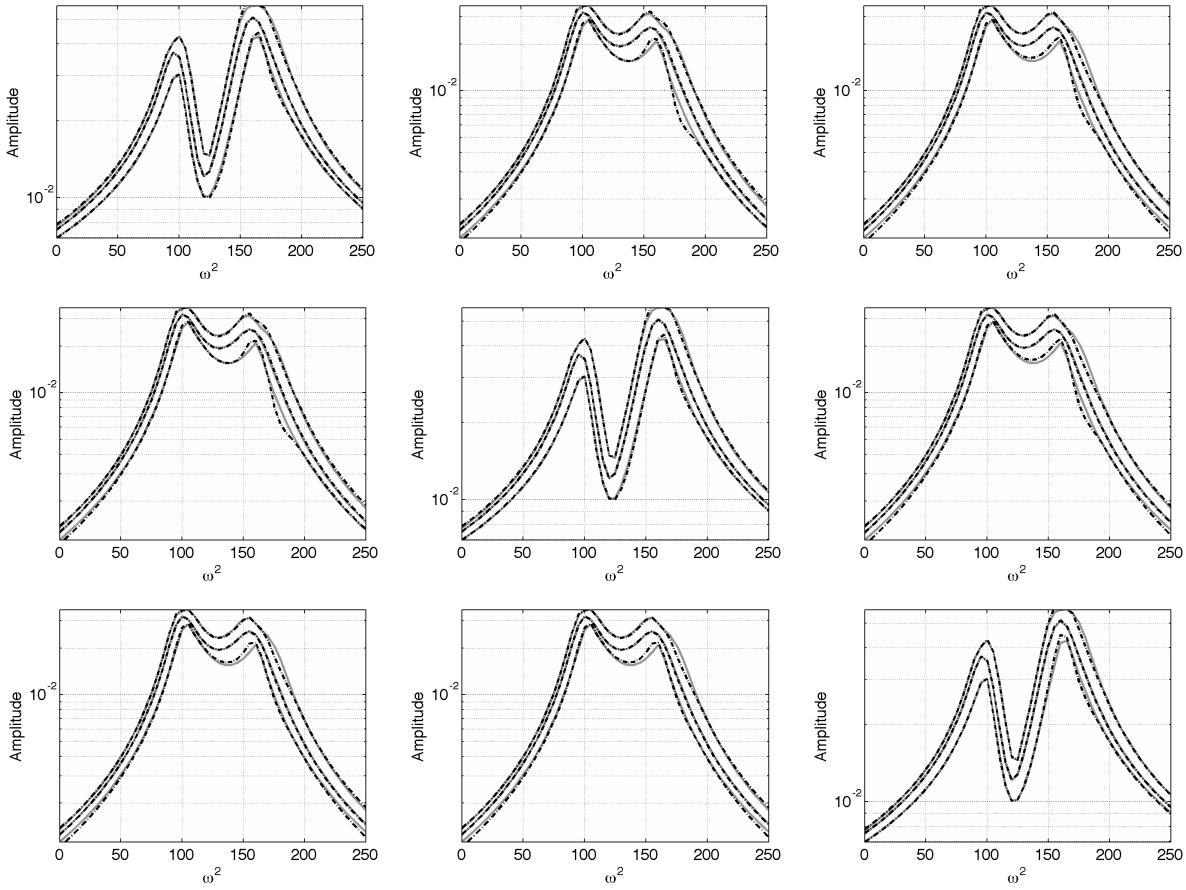

Figure 7: Frequency responses amplitudes of the three DoF system for the third application using the PC expansion up to the 4th order; Mean, 5% and 95 % lines of confidence interval are represented; From up to down is for the first to the last degree of freedom; From left-hand to right-hand is for an excitation applied from the first to the last degree of freedom; Thick grey lines are for MCS, dashed black lines are for PC expansion.

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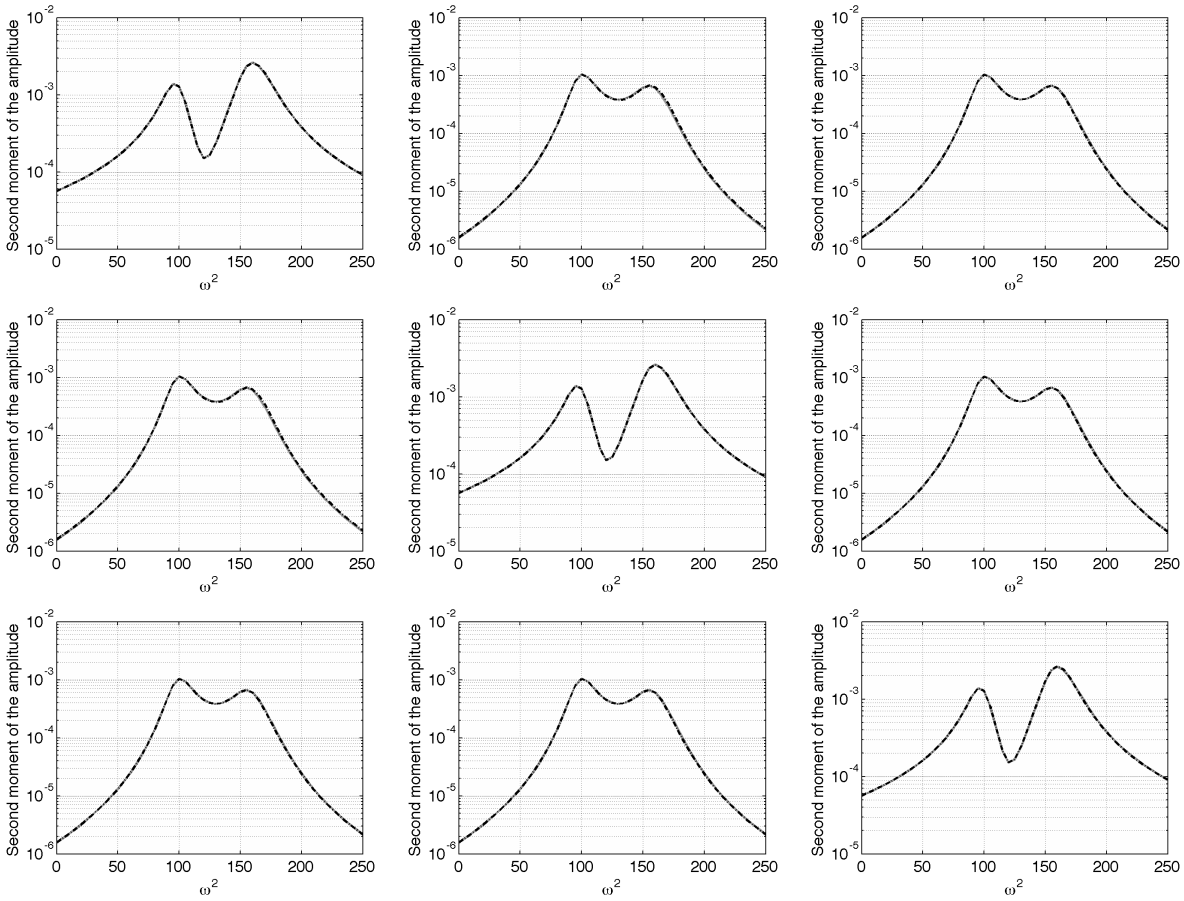

Figure 8: Second moment of frequency responses amplitudes of the three DoF system for the third application using the PC expansion up to the 4th order; From up to down is for the first to the last degree of freedom; From left-hand to right-hand is for an excitation applied from the first to the last degree of freedom; Thick grey lines are for MCS, dashed black lines are for PC expansion.

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computational effort similar to the one involved for stochastic linear static problems. Indeed, as for linear stochastic static problems, required expectations are limited to the product of three PC terms for all the computations steps while Newton-Raphson strategy and MCS are avoided.

Then, using the proposed strategies, the stochastic deflated inverse power method and the stochastic subspace inverse power method have a reduced computational effort for the construction of normal modes bases. Indeed, considering a stochastic application, the main step of these methods involves the resolution of a linear system which have the same size as the one produces by the static case. The two proposed methods can handle null eigenvalues, but the stochastic subspace inverse power method has the additional advantage to deal effectively also with repeated eigenvalues.

Three applications are studied to demonstrate the effectiveness of the proposed strategies. Comparisons of obtained results with the ones computed with MCS are satisfactory. In addition, it is found that frequency responses are more difficult to represent with PC than the stochastic normal modes.

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