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A NEW WAVELET PACKET BASES TO SOLVE FREDHOLM'S INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract. Different high order discretization methods and numerical expansions have been developed to find approximate solutions to integral equations with different kernels. However, direct application of standard numerical methods to the matrices obtained by discretizations of these equations can produce meaningless solutions. If the kernel is continuous, smooth and bounded, the integral operator is compact. In this context Fredholm's Integral Equations of the First Kind, i.e, $\mathcal{K}f(x) = \int_a^b h(x,y)f(y)dy = g(x)$, where f is the solution function and g is the data, are in general ill-conditioned inverse problems. However, restrictions to finite dimensional spaces where the unknown function f and g live can assure existence, unicity and well-conditioned of the problem. In this work we construct approximate solutions to inverse problems associated to integral operators of the first kind applying a wavelet decomposition. We restrict the problem to a bounded set of frequencies and we approximate the eigenfunctions of the operator from the images of finite set of wavelets functions trough the operator. Based on some properties of the basis, the resulting scheme is numerically stable.

1 INTRODUCTION

Integral equations of type:

$$\mathcal{K}f(x) = \int_a^b h(x,y)f(y)dy = g(x) \tag{1}$$

are called *Fredholm's Integral Equations of the First Kind* (IFK) and have been largely studied since Fourier times. Different high order discretization methods and numerical expansions have been developed to find approximate solutions to integral equations with singular kernels, see (Hao et al., 2014; Maleknejad et al., 2013; Serrano et al., 2014a; Du and Cui, 2008; Groestsch, 2007; Kress, 2014). However, direct application of standard numerical methods to the matrices obtained by discretizations of these equations can produce meaningless solutions (see Groestsch (2007) for a survey of the basic theory and methods). The quality of the solution to this type of equations largely depends on the functional spaces where f and g live.
If the kernel is continuous, smooth and bounded, the linear operator $\mathcal{K}$ is a compact operator and solving the integral equation is in general, an ill-conditioned problem. However, restrictions to finite dimensional spaces where the unknown function f and the data function g live can assure existence, unicity and well-conditioned of the problem, see Groestsch (2007).

The existence of eigenfunctions and eigenvalues of the integral operator $\mathcal{K}$ enables us to expande f, g or the kernel, in terms of the eigenfunctions. These expansions can be useful when trying to solve the integral equation $\mathcal{K}f = g$ and also we solving the Inverse Problem (IP), i. e., to recover f from g.
The existence of eigenfunctions of the operator $\mathcal{K}$ depends mostly on the symmetry of its kernel. But even in the case that they exist, computation of the eigenfunctions of a given operator could be a difficult task.

Wavelet decomposition has been widely applied to solve various integral equations because it provides fast computation schemes and, at the same time, it gives an appropiate representation of the functions involved, see (Serrano et al., 2014a,b, 2012; Walnut, 2002). In this work we construct approximate solutions to inverse problems associated to IFK applying wavelet decompositions. We restrict the problem to a bounded set of frequencies and approximate the eigenfunctions of the operator from the images of finite set of wavelets.

We consider a particular integral equations of the form (a particular case of IFK):

$$\mathcal{K}f(x) = \int_\Omega \widehat{h}(x,\omega)\widehat{f}(\omega)d\omega = g(x) \tag{2}$$

with Ω a compact set in the frequency domain.

If $\widehat{h}(x,\omega) = \widehat{h}(\omega)\, e^{ix\omega}$ and $\widehat{h}$ is a symmetric tempered distribution, we have a convolution operator

$$\mathcal{K}f(x) = \int_\Omega \widehat{h}(\omega)\widehat{f}(\omega)e^{ix\omega}\, d\omega = 2\pi\,(h * f)(x) = 2\pi \int_{\mathbb{R}} h(x-y)f(y)\, dy. \tag{3}$$

Littlewood Paley decomposition leads us to the partition:

$$\Omega = \bigcup_j \Omega_j \tag{4}$$

where

$$\Omega_j \simeq \{2^j\pi \leq |\omega| \leq 2^{j+1}\pi\} \tag{5}$$

is associated to the wavelet space W_j, see Walnut (2002) and Mallat (2009). We consider the subproblems related to Eq. 2:

$$\begin{aligned} \mathcal{K}_j &: L^2(\mathbb{R}) \to W_j, \\ \mathcal{K}_j\ \theta_j(x) &= \int_{\Omega_j} \widehat{h}(\omega)\widehat{\theta}_j(\omega)e^{ix\omega}d\omega \;=\; g_j(x) \end{aligned} \tag{6}$$

where g_j is the projection of g onto W_j. In general, the solution $\theta_j \notin W_j$, however we propose to approximate it in this space. Furthermore, we look for approximate eigenfunctions $\theta_{jk} \in W_j$ of the operator $\mathcal{K}_j$, such that:

$$\mathcal{K}_j\ \theta_{jk}(x) \;=\; \lambda_{jk}\theta_{jk}(x) + \varepsilon_{jk}(x) \tag{7}$$

with the minimum error $||\varepsilon_{jk}||_2$.

For these purposes, for each index j, we compute the approximate eigenfunctions $\theta_{jk} \in W_j$ from the images of a wavelet basis ψ_{jk} trough $\mathcal{K}_j$ for k in a finite set $\mathbb{K}_j \subset \mathbb{Z}$ in the context of a multiresolution analysis scheme, see Walnut (2002).

Using these elemental functions θ_{jk} we find approximate solutions to each subroblem (Eq. 6) and afterwards, we combine them to construct an approximate solution to the IP, (Eq. 2). Based on some properties of the basis, the resulting scheme is numerically stable.

In section 2 we define the approximate eigenfunctions. The approximate solution to the IP is developed in section 3. In section 4 we present an example. Finally we state the conclusions.

2 APPROXIMATE EIGENFUNCTIONS

2.1 The Wavelet Basis

In this work we choose a mother wavelet ψ well localized in both, time and frequency domain, that satisfies the following properties, see (Meyer, 1990):

1. the family $\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z}\}$ is an orthonomal basis of $L^2(\mathbb{R})$ associated to a multiresolution analysis (MRA),

2. $\psi \in \mathcal{S}$ (the Schwartz Clase) is a smooth, infinitely oscillating mother wavelet with fast decay.

3. the spectrum $|\widehat{\psi}(2^{-j}\omega)\,|$ is supported on the two-sided band
   $\Omega_j = \{\omega : 2^j(\pi - \alpha) \leq |\omega| \leq 2^{j+1}(\pi + \alpha)\}$, for some $0 < \alpha \leq \pi/3$.

The design of this basis and the implementation algorithm based on the Fast Fourier Transform have been developed by the authors in (Serrano and Fabio, 2011).

The properties of ψ ensure uniform convergence in each W_j.

2.2 Approximate Eigenfunctions in W_j

Looking forward to find accurate solutions to the IP, we construct approximate eigenfunctions θ_{jk} of the operator $\mathcal{K}$ to decompose the unknown function f. First, we define the images of the wavelete basis trough $\mathcal{K}$. They will be helpful to build the approximate eigenfunctions.

Definition 2.1 *Let*

$$v_{jk}(x) \;=\; \mathcal{K}_j\psi_{jk}(x) \;=\; \int_{\Omega_j} \widehat{h}(\omega)\widehat{\psi}_{jk}(\omega)e^{i\omega x}\,d\omega. \tag{8}$$

We remark that the functions v_{jk} are invariant under traslations and real since $\widehat{h}$ is a symmetric distribution:

$$v_{jk}(x) = v_{j0}(x - 2^{-j}k).$$

Let $\mathbb{K}$ any finite index set $\{k_1, \ldots, k_M\}$ and denote

$$W_j(\mathbb{K}) = span\{\psi_{jk}, k \in \mathbb{K}\} \subset W_j.$$

Next we define the approximate eigenfunction.

Definition 2.2 *A function θ is an approximate eigenfunction of $\mathcal{K}$ in the wavelet space W_j, if $\theta \in W_j(\mathbb{K})$ for some finite set $\mathbb{K}$, $\|\theta\|_2 = 1$ and there is $\lambda \in \mathbb{R}$ such that:*

$$\langle \mathcal{K}_j\ \theta - \lambda\theta, \psi_{jk} \rangle \;\; = \;\; 0 \tag{9}$$

for all $k \in \mathbb{K}$.

In order to find θ we observe that from

$$\theta(x) = \sum_{k\in\mathbb{K}} c_{jk}\psi_{jk}(x) \tag{10}$$

it follows:

$$\mathcal{K}_j\theta(x) - \lambda\theta(x) = \sum_{k\in\mathbb{K}} c_{jk}v_{jk}(x) - \lambda \sum_{k\in\mathbb{K}} c_{jk}\psi_{jk}(x) \tag{11}$$

Then, Eq. 9 becomes:

$$\sum_{k\in\mathbb{K}} c_{jk}\langle v_{jk}, \psi_{jn}\rangle - \lambda c_{jn} \;\; = \;\; 0 \;\text{ for } n \in \mathbb{K} \tag{12}$$

This system is equivalent to the spectral problem in $\mathbb{R}$

$$GC \;\; = \;\; \lambda C \tag{13}$$

where $C \in \mathbb{R}^{M\times 1}$ and $G = \left(\langle v_{jk}, \psi_{jn}\rangle\right)_{k,n\in\mathbb{K}} \in \mathbb{R}^{M\times M}$.

We assume that $\widehat{h}(\omega)$ is a real symmetric distribution and there exist positive constants A, B such that

$$0 < A \leq |\widehat{h}(\omega)| \leq B < \infty.$$

Then we can prove that the matrix G is symmetric, Toepliz and non singular.

Now, let $\lambda_m \in \mathbb{R}$ an eigenvalue with normalized eigenvector $C^{(m)} = \left(c_{jk}^{(m)}\right)_{k\in\mathbb{K}}$. Then, the function

$$\theta^{(m)}(x) \;\; = \;\; \sum_{k\in\mathbb{K}} c_{jk}^{(m)}\psi_{jk}(x) \tag{14}$$

is an approximate eigenfunction in W_j with eigenvalue λ_m.

In the same way, we complete an orthonormal basis of eigenvectors $\{C^{(1)}, \ldots, C^{(M)}\}$ and, hence, an orthonormal basis for $W_j(\mathbb{K})$ of approximated eigenfunctions $\{\theta^{(1)}(x), \ldots, \theta^{(M)}(x)\}$.

Note that $Q = \left(C^{(1)}, \ldots, C^{(M)}\right) \in \mathbb{R}^{M\times M}$ is an orthogonal matrix and it gives the coordinates of the new basis in the wavelet basis $\{\psi_{jk_1}(x), \ldots, \psi_{jk_M}(x)\}$ of $W_j(\mathbb{K})$.

This scheme can be extended to W_j through a partition of $\mathbb{Z}$ of disjoint finite sets $\mathbb{K}_l$. Then

$$W_j = \bigoplus_l W_j(\mathbb{K}_l) \tag{15}$$

and the union of the respective bases give us an orthonormal basis of W_j.

We will call it *Approximate Eigenfunctios Wavelet Packet Bases* or *AEWP- Bases*.

3 APPROXIMATE SOLUTION TO THE IP

Suppose $g \in L^2(\mathbb{R})$ and consider the IP given in Eq. 2

$$g(x) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \langle g, \psi_{jk}\rangle \psi_{jk}(x). \tag{16}$$

For each j let

$$g_j(x) = \sum_{j\in\mathbb{K}} \langle g, \psi_{jk}\rangle \psi_{jk}(x) \tag{17}$$

an approximate projection of g on W_j, where $\mathbb{K} = \mathbb{K}(j)$ is some finite index set such that

$$||g_j||^2 = \sum_{k\in\mathbb{K}} |\langle g, \psi_{jk}\rangle|^2 = (1-\rho)^2 \sum_{k\in\mathbb{Z}} |\langle g, \psi_{jk}\rangle|^2 \tag{18}$$

for some $\rho \ll 1$.

Denote $\{\theta_j^{(1)}(x), \ldots, \theta_j^{(M)}(x)\}$ the approximate eigenfunctions basis of $W_j(\mathbb{K})$ and the coordinates matrix $Q_j \in \mathbb{R}^{M\times M}$.

The coefficients $d_j^{(m)}$ of g_j are obtained from its wavelet coefficients by

$$\left(d_j^{(m)}\right) = Q^t \left(\langle g_j, \psi_{jk}\rangle\right)_{k\in\mathbb{K}}. \tag{19}$$

Recalling that the eigenvalues λ_m are non zero, finally we obtain the approximate solution in $W_j(\mathbb{K})$,

$$f_j(x) = \sum_{m\in\mathbb{K}} \frac{d_j^{(m)}}{\lambda_m} \theta_j^{(m)}(x). \tag{20}$$

4 EXAMPLE

An important class of IP are the one associated to pseudodifferential operator with kernel $\widehat{h}(\omega) = \frac{1}{(1+\omega^2)^{(s/2)}}$. They satisfy the hypotheses to ensure the existence of approximate eigenfunctions bases.

In this section we consider de FIE related to Eq. 2 with $s = 0.3$. The subproblems are restrict to a bounded set of frequencies $\Omega_j = \{\omega : 2^{j+1}(\pi/3) \leq |\omega| \leq 2^{j+3}(\pi/3)\}$.

Figures 1(a) and 1(b) show the graph of $\psi_{-1,0}(x)$ and $\upsilon_{-1,0}(x)$, respectively. The computed of $\theta_{-1,4}^{(5)}(x)$ and $\theta_{-1,4}^{(9)}(x)$ are displayed in Figures 2(a) and 2(b), respectively.

(Hao et al., 2014)

5 CONCLUSIONS

In this work we construct approximate solutions to inverse problems associated to integral operators of the first kind applying wavelet decompositions. We restrict the problem to a bounded set of frequencies and we approximate the eigenfunctions of the operator from the images of finite set of wavelets functions trough the operator. The approximate eigenfunctions are defined for the operator and a given wavelet basis. The advantage of the proposed scheme is that the coefficients of the decomposition of the solution on the approximate eigenfunctions can be easily be computed from the coefficients of the data on the wavelet basis. The error depends on the dimension of $W_j(\mathbb{K})$.

The perspective of this line of research leads us to extend this technique to operators whose kernel $\widehat{h}(x,\omega)$ depends both in x and ω, leading to a broad class of problems.

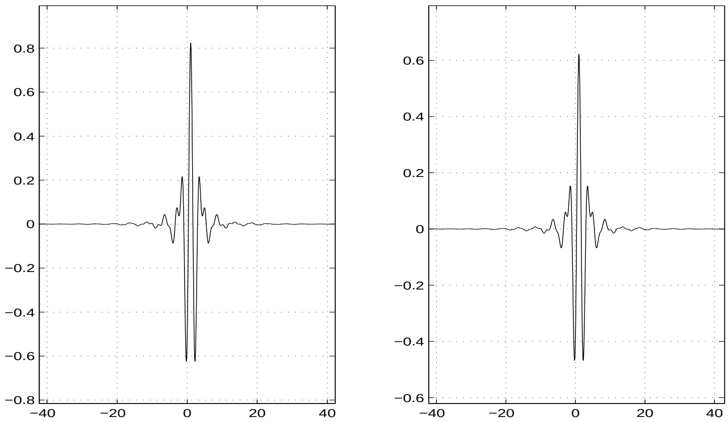

Figure 1: (a) $\psi_{-1,0}(x)$, (b) $\upsilon_{-1,0}(x)$

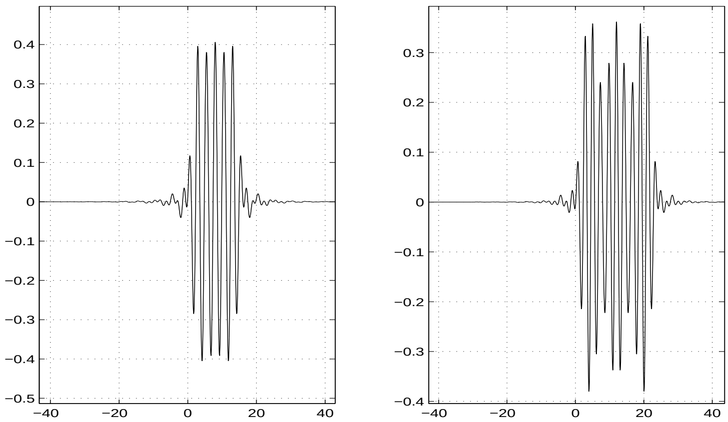

Figure 2: (a) $\theta^{(5)}_{-1,4}(x)$, (b) $\theta^{(9)}_{-1,4}(x)$

REFERENCES

Du H. and Cui M. Approximate solution of the fredholm integral equation of the first kind in a reproducing kernel hilbert space. *Applied Methematics Letters*, 21:617–623, 2008.

Groestsch C. Integral equations of the first kind, inverse problems and regularization: a crash course. *Journal of Physiscs, Conference Series*, 73, 2007.

Hao S., Barnett A., Martinsson P., and Young P. High order accurate methods for nystrom discretization of integral equations on smooth curves in the plane. *Adv. Computational mathematics*, 40:245–272, 2014.

Kress R. *Linear Integral Equations. Applied Mathematical Sciences*. Springer, 2014.

Maleknejad K., Nosrati M., and Njafi E. Wavelet galerkin method for solving singular integarl equations. *American applied sciences*, 7:3847–3852, 2013.

Mallat S. *A Wavelet Tour of Signal Processing*. S. Mallat, 2009.

Meyer Y. Ondelettes et operateurs 2. operatteurs de calderon zygmund. *Hermann et Cie*, 1990.

Serrano E. and Fabio M. Sub-wavelets: una nueva familia de funciones elementales en el contexto de un análisis de multirresolución. *Actas III MACI*, 2011.

Serrano E., Troparevsky M., and Fabio M. Wavelet-vaguelet decomposition methods to solve pseudodifferential inverse problems. *Proceedings SIMMAC*, 2012.

Serrano E., Troparevsky M., and Fabio M. Solving deconvolution type problems by wavelet decomposition methods. *Journal of Contemporary Mathematical Analysis*, 49, 2014a.

Serrano E., Troparevsky M., and Fabio M. Wavelet projection methods for solving pseudodifferential inverse problems. *Internaional Journal of Waveletes Multiresolution and Information Processing*, 12:14500251–17, 2014b.

Walnut D.F. *An Introduction to Wavelet Analysis. Applied and Numerical Harmonic Analysis*. Birkhauser, 2002.