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LINEAR CURVED BEAM FINITE ELEMENT WITH EXACT RIGID BODY DISPLACEMENTS

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Abstract. Finite element models of curved beams using polynomial interpolation functions for discretization of radial and tangential displacement fields show ill-conditioning problems and shear locking. Polynomial and Fourier trigonometric-function elements for in-plane vibration of thin and thick curved beams have been proposed by different authors as enriching functions to avoid the ill-conditioning problem mentioned, avoiding membrane and shear locking. The research presented in this paper focuses on the development of finite elements for linear analysis of thin and thick beams that include exact rigid body motion interpolating functions. The dynamic behavior of thin and thick beams is analyzed using the proposed finite-element discretization. Modal analysis of clamped-clampled circular arches is carried out as numerical examples. Precision obtained with the proposed finite element is compared with traditional finite elements and other numerical results published on these models.

1 INTRODUCCION

Elastic curved beams in small deformations have received significant attention in the literature during last decades. In the displacement-based finite element (FE) method, the classical polynomials used to approximate displacement fields of structural members are chosen to satisfy the requirements of state of constant strain and the rigid-body motion representation.

Timoshenko and Euler-Bernoulli beam theory with classical low order polynomial interpolation functions do not provide very accurate results in curved beam elements, and are prone to shear and membrane locking (Leung and Zhu, 2004). Because low-order interpolation function finite elements cannot satisfy the requirements of rigid-body motion representation, different strategies have been proposed in the literature. High order polynomials have been proposed (Petyt and Fleischer, 1971; Babu and Prathap, 1986). A two-node 8-degree-of-freedom (dof) cubic linear element including shear deformation and rotatory inertia was proposed to eliminate locking (Krishnan and Suresh, 1998. Fourier p-elements for in-plane vibration of thin and thick curved beams have been proposed (Leung and Zhu 2004), using Fourier trigonometric functions as enriching functions to avoid the ill-conditioning problems associated with high order polynomials in thin and thick curved beams. Fourier shape functions proposed by are defined to be zero at extreme nodes of the finite element (Leung and Chang, 1998). This definition allows the enrichment of the finite element with additional internal degrees of freedom (additional generalized coordinates) of the FE with no change in the inter-element displacement compatibility. These authors proved that with additional Fourier degrees of freedom, the accuracy of the computed natural frequencies is greatly improved (Leung and Chang, 1998; Leung and Zhu, 2004).

The FE proposed in this paper includes specific interpolation functions that create the subspace that solves the exact rigid body displacement fields of the thin or the thick curved beam models in small displacement theory. In addition, deformation coordinates and interpolation functions are proposed to provide quadratic or linear dependence of deformation fields (elongation, shear, and curvature) as functions of the spatial coordinate of the element. The strategy avoids the need of significant enlargement of the number of degrees of freedom of the FE and provides exact representation of rigid body displacements.

The paper is organized as follows. Firstly, the proposed thin curved beam FE is presented. The small displacement linear kinematic model is defined, from which, the exact rigid displacement subspace displacement fields and the displacement subspace associated with quadratic elongation and curvature are obtained. Mass and stiffness matrices of the proposed FE are developed computing closed-form expressions for the nine generalized coordinates used for the kinematic representation. Linear kinematic restrictions are imposed to the generalized coordinates of the structural model to ensure displacement compatibility between adjacent finite elements and geometric boundary conditions. The order reduction of the structural model to a set of independent generalized coordinates is described and used for natural frequencies and mode shapes computation of the restrained MK linear model developed using the proposed FE. Secondly, an analogous strategy is applied to thick curved beams, developing a FE with eleven generalized coordinates that can represent exact rigid body displacements, quadratic elongation and shear deformation, and linear curvature. Finally, accuracy of the proposed FE is evaluated comparing natural frequency and mode shape estimation of the corresponding continuum model for simple curved beam structures.

2 THIN CURVED BEAM

The structural element under analysis in this section is a thin curved beam of constant radius of curvature R in the undeformed configuration subjected to small deformations. The ratio of sectional radius of gyration to radius of curvature of the beam, $\frac{r_g}{R} \ll 1$ and $\frac{r_g}{\theta_o R} \ll 1$, where θ_o is the total angle of the curved beam.

As shown in Figure 1, displacements fields that describe the deformed configuration are the tangential $u_\theta(s,t)$ and radial $u_r(s,t)$ displacements, where s is the spatial coordinate that defines the location of every arch section in the undeformed configuration.

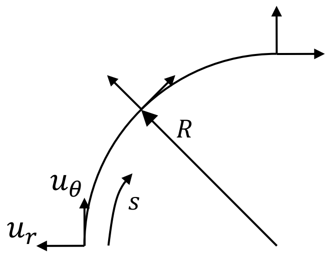

Figure 1. Thin curved beam model and displacement fields.

2.1 Kinematic model for FE of thin curved beam

Rigid body displacement field in a thin curved beam

Because the use of conventional finite-order polynomial functions cannot represent rigid body motion of the curved beam, interpolating functions that satisfy rigid body displacements are considered along with interpolating functions that provide quadratic elongation (extensional strain) and curvature in the thin beam.

According to the classical thin shell theory, the extensional strain ε, the sectional rotation φ, and the curvature κ, can be expressed for small tangential and radial displacements by the following linear differential operators on the displacement fields:

$$\varepsilon(s,t) = \frac{\partial u_\theta(s,t)}{\partial s} + \frac{u_r(s,t)}{R} \tag{1}$$

$$\varphi(s,t) = \frac{u_\theta(s,t)}{R} - \frac{\partial u_r(s,t)}{\partial s} \tag{2}$$

$$\kappa(s,t) = \frac{1}{R}\frac{\partial u_\theta(s,t)}{\partial s} - \frac{\partial^2 u_r(s,t)}{\partial s^2} \tag{3}$$

To compute rigid body fields, ${}_{rb}u_r(s)$ and ${}_{rb}u_\theta(s,t)$, compatible with this linear kinematic model we can solve for the radial and tangential fields that produce zero strain and zero curvature in the thin beam:

$$\frac{d\,{}_{rb}u_\theta(s,t)}{ds} + \frac{{}_{rb}u_r(s)}{R} = 0 \tag{4}$$

$$\frac{1}{R}\frac{d\,{}_{rb}u_\theta(s,t)}{ds} - \frac{d^2\,{}_{rb}u_r(s)}{ds^2} = 0 \tag{5}$$

From Eq.(4),

$$\frac{d\,{}_{rb}u_\theta(s,t)}{ds} = -\frac{{}_{rb}u_r(s)}{R} \tag{6}$$

Substituting Eq.(6) in Eq.(5)

$$-\frac{1}{R}\frac{{}_{rb}u_r(s)}{R}-\frac{d^2{}_{rb}u_r(s)}{ds^2}=0 \tag{7}$$

Therefore, a rigid displacement field satisfies

$$\frac{d^2{}_{rb}u_r(s)}{ds^2}+\frac{1}{R^2}{}_{rb}u_r(s)=0 \tag{8}$$

The solution of this linear differential equation is

$${}_{rb}u_r(s)=\sin\left(\frac{s}{R}\right)u_1+\cos\left(\frac{s}{R}\right)u_2 \tag{9}$$

Where u_1 and u_2 are arbitrary constants (static problems) or functions of time (dynamic problems).

From Eqs.(9) and (6)

$$\frac{d_{rb}u_\theta(s,t)}{ds}=-\frac{1}{\mathrm{R}}\sin\left(\frac{s}{R}\right)u_1-\frac{1}{\mathrm{R}}\cos\left(\frac{s}{R}\right)u_2 \tag{10}$$

Integrating this equation, we solve for the tangential displacement field consistent with rigid body displacements of the curved beam

$${}_{rb}u_\theta(s,t)=\cos\left(\frac{s}{R}\right)u_1-\sin\left(\frac{s}{R}\right)u_2+1\,u_3 \tag{11}$$

Equations (9) and (11) represent rigid body displacements fields; u_1, u_2 and u_3, generalized coordinates to represent rigid body displacement in the proposed thin curved-beam finite element, associated to the following interpolating functions

$$\psi_{u_r1}(s)=\sin\left(\frac{s}{R}\right) \tag{11a}$$
$$\psi_{u_r2}(s)=\cos\left(\frac{s}{R}\right) \tag{11b}$$
$$\psi_{u_r3}(s)=0 \tag{11c}$$
$$\psi_{u_\theta1}(s)=\cos\left(\frac{s}{R}\right) \tag{11d}$$
$$\psi_{u_\theta2}(s)=-\sin\left(\frac{s}{R}\right) \tag{11e}$$
$$\psi_{u_\theta3}(s)=1 \tag{11f}$$

To complete the displacement fields to formulate the finite element, conventional polynomial interpolating functions could be added. Instead, a quadratic deformation condition is assumed for elongation and curvature in the proposed FE, as indicated in the following equations:

$$\varepsilon(s,t)=\frac{\partial u_\theta(s,t)}{\partial s}+\frac{u_r(s,t)}{R}=\frac{1}{L}u_4+u_5\frac{s}{L^2}+u_6\frac{s^2}{L^3} \tag{12}$$

$$\kappa(s,t)=\frac{1}{R}\frac{\partial u_\theta(s,t)}{\partial s}-\frac{\partial^2u_r(s,t)}{\partial s^2}=\frac{1}{R^2}u_7+\frac{s}{R^3}u_8+\frac{s^2}{R^3}u_9 \tag{13}$$

All generalized coordinates, u_j, are defined with displacement units. For this reason the geometric parameters L and R of the FE element are used in the definition of these deformation fields.

Consistent with the assumed deformation fields, the following displacement fields can be computed:

$${}_{ld}u_r(s,t)=c_0+c_1s+c_2s^2 \tag{14a}$$

$$ {}_{ld}u_\theta(s,t) = d_1 s + d_2 s^2 + d_3 s^3 \tag{14b}$$

Replacing Eqs. (14) in Eq.(12) and in Eq. (13):

$$ d_1 + 2d_2 s + 3d_3 s^2 + \frac{c_0}{R} + \frac{c_1}{R} s + \frac{c_2}{R} s^2 = \frac{1}{L} u_4 + u_5 \frac{s}{L^2} + u_6 \frac{s^2}{L^3} \tag{15a}$$

$$ \frac{1}{R}(d_1 + 2d_2 s + 3d_3 s^2) - 2c_2 = \frac{1}{R^2} u_7 + \frac{s}{R^3} u_8 + \frac{s^2}{R^4} u_9 \tag{15b}$$

Equating terms of s of the same power

$$\begin{bmatrix} 1/R & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/R & 0 & 0 & 2 & 0 \\ 0 & 0 & 1/R & 0 & 0 & 3 \\ 0 & 0 & -2 & 1/R & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/R & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/R \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{L} u_4 \\ \frac{u_5}{L^2} \\ \frac{u_6}{L^3} \\ \frac{1}{R^2} u_7 \\ \frac{1}{R^3} u_8 \\ \frac{1}{R^4} u_9 \end{bmatrix} \tag{16}$$

Solving for the interpolation-function coefficients:

$$ d_3 = \frac{1}{3R^3} u_9 \tag{17a}$$

$$ d_2 = \frac{1}{2R^2} u_8 \tag{17b}$$

$$ c_1 = \frac{Ru_5}{L^2} - \frac{1}{R} u_8 \tag{17c}$$

$$ c_2 = \frac{Ru_6}{L^3} - \frac{1}{R^2} u_9 \tag{17d}$$

$$ d_1 = \frac{1}{R} u_7 + \frac{2R^2 u_6}{L^3} - \frac{2}{R} u_9 \tag{17e}$$

$$ c_0 = \frac{R}{L} u_4 - u_7 - 2\frac{R^3 u_6}{L^3} + 2\, u_9 \tag{17f}$$

Finally then, summing the rigid-body displacement fields and the linear-deformation fields, the displacement functions for the proposed FE are

$$ u_r(s,t) = \sin\left(\frac{s}{R}\right) u_1 + \cos\left(\frac{s}{R}\right) u_2 + 0\, u_3 + \frac{R}{L} u_4 + \frac{R}{L^2} s\, u_5 + \left(-2\frac{R^3}{L^3} + \frac{Rs^2}{L^3}\right) u_6 + (-1) u_7 + \left(-\frac{s}{R}\right) u_8 + \left(2 - \frac{s^2}{R^2}\right) u_9 \tag{18a}$$

$$ u_\theta(s,t) = \cos\left(\frac{s}{R}\right) u_1 - \sin\left(\frac{s}{R}\right) u_2 + u_3 + 0\, u_4 + 0\, u_5 + \frac{2R^2 s}{L^3} u_6 + \frac{s}{R} u_7 + \left(\frac{s^2}{2R^2}\right) u_8 + \left(-\frac{2s}{R} + \frac{s^3}{3R^3}\right) u_9 \tag{18b}$$

Compatibility conditions between thin curved beam elements require the satisfaction of radial displacement, tangential displacement and rotation continuity. Rotation can be computed using Eq. (2) for the prescribed displacement fields, where for the proposed radial displacement field

$$ \frac{\partial u_r(s,t)}{\partial s} = \frac{1}{R}\cos\left(\frac{s}{R}\right) u_1 - \frac{1}{\mathrm{R}}\sin\left(\frac{s}{R}\right) u_2 + 0\, u_3 + 0 u_4 + \frac{R}{L^2} u_5 + \left(\frac{2Rs}{L^3}\right) u_6 + 0 u_7 + \left(-\frac{1}{R}\right) u_8 + \left(-\frac{2s}{R^2}\right) u_9 \tag{19}$$

Once the kinematic model has been defined allowing the displacement fields to represent rigid body displacements within the linear kinematic assumption used for deformation computation and quadratic deformation fields, the corresponding mass and stiffness matrices of the finite element model can be computed for the nine generalized coordinates used in the formulation of the FE (three coordinates for rigid body displacement in the plane and six coordinates for longitudinal and flexural deformation) by standard integration.

Because the proposed formulation does not use nodal displacements coordinates (such as those associated to standard polynomial FE shape functions) that automatically guarantee displacement and rotation compatibility, the proposed FE formulation requires the inclusion of kinematic constraint equations to force displacement and rotation compatibility of boundary nodes of adjacent elements. This aspect is presented after element stiffness and mass matrices are computed for the proposed generalized coordinates in the following section.

2.2 Potential and kinetic energy representation

The potential energy of the thin curved beam is expressed as

$$U = \frac{1}{2}\int_0^l (EA\varepsilon^2 + EI\kappa^2)ds \tag{20}$$

where E is the Young modulus, A is the cross sectional area, and I is the second moment of inertia of the cross sectional area. In this model, shear deformation is neglected.

Substituting the expressions of longitudinal strain and curvature

$$U_e = \int_0^l (EA(\frac{\partial u_\theta(s,t)}{\partial s} + \frac{u_r(s,t)}{R})^2 + EI(\frac{1}{R}\frac{\partial u_\theta(s,t)}{\partial s} - \frac{\partial^2 u_r(s,t)}{\partial s^2})^2)ds \tag{21}$$

Neglecting rotational inertia of the thin curved beam, the kinetic energy can be expressed with the following alternative expressions:

$$T = \frac{1}{2}\int_0^l (\rho A\frac{\partial u_r}{\partial t}^2 + \rho A\frac{\partial u_\theta}{\partial t}^2)ds \tag{22a}$$

$$T = \frac{1}{2}\int_0^l (\rho A\frac{\partial u_r}{\partial t}^2 + \rho A\frac{\partial u_\theta}{\partial t}^2)ds + \frac{1}{2}\int_0^l \rho I\frac{\partial \varphi}{\partial t}^2 ds \tag{22b}$$

where ρ is de density of the material. Equation 22b is the kinetic energy of the model if rotational inertia is included.

Replacing the displacement fields and rotation field by the linear combination of the interpolating functions times the FE generalized coordinates given in Eqs. (18a) and (18b), the stiffness and mass matrices can be computed. The elements of the stiffness and mass matrices can be expressed in its general form as

$$_eK_{ij} = {}_eK_{ji} = \frac{\partial^2 U_e}{\partial u_i \partial u_j} \tag{23a}$$

$$_eM_{ij} = {}_eM_{ji} = \frac{\partial^2 T}{\partial \dot{u}_i \partial \dot{u}_j} \tag{23b}$$

where $_e\mathbf{K}$ and $_e\mathbf{M}$ are eigth by eight FE matrices.

Closed form expressions are computed for the elements of the stiffness and mass matrices (which are not included for brevity) in the case of constant parameter models (E, A, I, ρ). In the case of arbitrarily varying geometric or mechanical parameters, numerical integration is required for mass and stiffness matrices.

2.3 Dynamic model and kinematic constraints

Given a structural model developed with the proposed FE as that shown in Figure 2, the equations of motion of the model in free vibration can be expressed as

$$\mathbf{M}\ddot{\boldsymbol{r}} + \mathbf{K}\boldsymbol{r} = \mathbf{0} \tag{24}$$

$$\boldsymbol{h}(\boldsymbol{r}) = \boldsymbol{L}\,\boldsymbol{r} = \mathbf{0} \tag{25}$$

Where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices of the model in generalized coordinates defined in column vector $\boldsymbol{r}$; $\boldsymbol{h}(\boldsymbol{r})$ is the vector of kinematic constraints associated to the imposed boundary conditions of the structure and the kinematic compatibility conditions imposed by continuity of radial and tangential displacement fields and rotation between adjacent elements. These linear equations impose displacement compatibility between finite elements used in the discretized structural model.

Because the model considers eight degrees of freedom for each FE and none of the generalized displacements in the radial or the tangential direction are shared between elements (not even adjacent elements) in the proposed formulation, $\mathbf{M}$ and $\mathbf{K}$ are block diagonal matrices of size $8N_e \times 8N_e$,

$$\mathbf{K} = \mathrm{diag}({}_e^1\mathbf{K},\ {}_e^2\mathbf{K},\ {}_e^3\mathbf{K}, \dots, {}_e^{N_e}\mathbf{K}) \tag{26}$$

$$\mathbf{M} = \mathrm{diag}({}_e^1\mathbf{M},\ {}_e^2\mathbf{M},\ {}_e^3\mathbf{M}, \dots, {}_e^{N_e}\mathbf{M}) \tag{27}$$

where ${}_e^j\mathbf{K}$ and ${}_e^j\mathbf{M}$ are the proposed FE stiffness and mass matrices.

To illustrate in a specific case the assembly of restraint matrix $\boldsymbol{L}$, let us consider a model of a thin curved beam clamped in one end and free in the other, with two finite elements of the type proposed (see Figure 2).

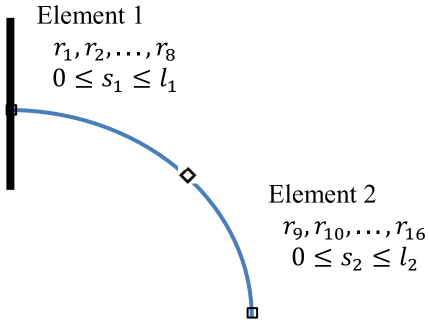

Figure 2. Two-element model of thin curved beam with proposed FE

The first constraint equation on the generalized coordinates $\boldsymbol{r}$ is given by radial displacement field in element 1 equal to zero at $s_1 = 0$ (clamped end)

$$h_1(\boldsymbol{r}) = {}_1u_r(0,t) = N_1(0)r_1 + 0\,r_2 + N_2(0)r_3 + 0\,r_4 + N_3(0)r_5 + 0\,r_6 + \sin\left(\frac{0}{R}\right)r_7 + \cos\left(\frac{0}{R}\right)r_8 = 0 \tag{28}$$

The second constraint equation is given by tangential displacement field in element 1 equal to zero at $s_1 = 0$ (clamped end)

$$h_2(\boldsymbol{r}) = {}_1u_\theta(0,t) = 0\,r_1 + N_1(0)r_2 + 0\,r_3 + N_2(0)r_4 + 0\,r_5 + N_3(0)r_6 + \cos\left(\frac{0}{R}\right)r_7 - \sin\left(\frac{0}{R}\right)r_8 = 0 \tag{29}$$

The third constraint at the clamped end, is rotation equal to zero at $s_1 = 0$

$$h_3(\boldsymbol{r}) = {}_1\varphi(0,t) = 0 \tag{30}$$

Using Eqs. (2), (18) and (19), this constraint can be written in terms of the generalized coordinates of the first FE. Defining as $r_8, r_9, r_{10}, \dots, r_{16}$ the generalized coordinates associated to FE deformation coordinates $u_1, u_2, u_3, \dots, u_8$ of the second FE, the sixth, seventh and eight constraints imposed by displacement fields compatibility between adjacent elements 1 and 2 can be expressed as

$$h_4(\boldsymbol{r}) = \ {}_2u_r(0,t) - \ {}_1u_r(l_1,t) = 0 \tag{31a}$$

$$h_5(\boldsymbol{r}) = \ {}_2u_\theta(0,t) - \ {}_1u_\theta(l_1,t) = 0 \tag{31b}$$

$$h_6(\boldsymbol{r}) = \ {}_2\varphi(0,t) - \ {}_1\varphi(l_1,t) = 0 \tag{31c}$$

Replacing the displacement and rotation fields as functions of the generalized coordinates and interpolating functions proposed, the six constraint equations given in Eqs. (28) through (31) can be expressed in the form of Eq. (25) for this example.

The dynamic analysis can be done reducing the generalized coordinates to an independent set of coordinates $\boldsymbol{q}$, with a number of degrees of freedom N_q

$$N_q = N_r - N_h \tag{32}$$

where N_h is the number of constraints on the model coordinates. In the example considered (model in Fig. 2), $N_q = 16 - 6 = 10$ dofs.

Selecting $\boldsymbol{q}$ as an unconstraint subset of generalized coordinates $\boldsymbol{r}$, we can express

$$\boldsymbol{r}(t) = \boldsymbol{L_{rq}}\, \boldsymbol{q}(t) \tag{33}$$

solving for the slave displacements as linear functions of the unconstrained generalized coordinates $\boldsymbol{q}$, using Eq. (25) (Inaudi, 2010).

The unconstrained reduced order model is finally expressed as

$$\mathbf{M}_{\boldsymbol{q}}\ddot{\boldsymbol{q}}(t) + \mathbf{K}_{\boldsymbol{q}}\boldsymbol{q}(t) = \mathbf{0} \tag{34}$$

where

$$\mathbf{M}_{\boldsymbol{q}} = \boldsymbol{L_{rq}}^T \mathbf{M}\, \boldsymbol{L_{rq}} \tag{35a}$$

$$\mathbf{K}_{\boldsymbol{q}} = \boldsymbol{L_{rq}}^T \mathbf{K}\, \boldsymbol{L_{rq}} \tag{35b}$$

Using the reduced-order model with independent coordinates, natural frequencies and modes of vibration can be computed using standard eigenvalue-problem solvers.

If the model had non conservative external loads $\boldsymbol{F_r}$ assembled by virtual work on the generalized coordinates $\boldsymbol{r}$, the reduced-order model subjected to external loading take the form

$$\mathbf{M}_{\boldsymbol{q}}\ddot{\boldsymbol{q}}(t) + \mathbf{K}_{\boldsymbol{q}}\boldsymbol{q}(t) = \boldsymbol{L_{rq}}^T \boldsymbol{F_r}(t) \tag{36}$$

In section 4 numerical results are shown for static and dynamic analysis of structural models of thin arches and beams in planar deformation.

3 THICK CURVED BEAM MODEL

The structural element under analysis in this section is a thick curved beam of constant radius of curvature R in the undeformed configuration subjected to small deformations.

3.1 Kinematic model for FE of thick curved beam

As shown in Figure 3, the displacements fields that describe the deformed configuration are the tangential displacement $u_\theta(s,t)$, the radial displacement $u_r(s,t)$ and the rotation field $\alpha(s,t)$, where s is the spatial coordinate that defines the location of every arch section in the undeformed configuration. This description allows for the representation of shear deformation in the beam.

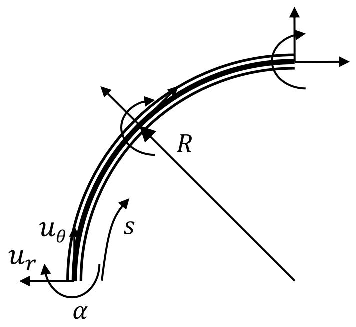

Figure 3. Thick curved beam

For a thick beam with shear deformations, the extensional strain ε, the shear deformation γ and the curvature κ, can be expressed for small tangential displacements, radial displacements and rotations, by the following linear differential operators

$$\varepsilon(s,t) = \frac{\partial u_\theta(s,t)}{\partial s} + \frac{u_r(s,t)}{R} \tag{37a}$$

$$\gamma(s,t) = \alpha(s,t) + \frac{\partial u_r(s,t)}{\partial s} - \frac{u_\theta(s,t)}{R} \tag{37b}$$

$$\kappa(s,t) = \frac{\partial \alpha(s,t)}{\partial s} \tag{37c}$$

Rigid body displacement field

Because conventional polynomial interpolation functions cannot represent rigid body motion of the thick curved beam, special interpolating functions that satisfy rigid body displacements are included in the kinematic model of the proposed FE. To compute rigid body fields compatible with this linear kinematic model we can solve for the radial, tangential and rotational fields that produce zero strain, zero curvature and zero shear deformation in the thick curved beam:

$$\frac{du_\theta(s,t)}{ds} + \frac{u_r(s,t)}{R} = 0 \tag{38a}$$

$$\alpha(s,t) + \frac{du_r(s,t)}{ds} - \frac{u_\theta(s,t)}{R} = 0 \tag{38b}$$

$$\frac{\partial \alpha(s,t)}{\partial s} = 0 \tag{38c}$$

Integrating Eq. (38c)

$$_{rb}\alpha(s,t) = u_1 \tag{39}$$

Differentiating Eq. (38a)

$$\frac{d^2u_r(s,t)}{ds^2} - \frac{du_\theta(s,t)}{ds}\frac{1}{R} = 0 \tag{40}$$

From Eqs. (38a) and (40)

$$\frac{d^2u_r(s,t)}{ds^2} + \frac{1}{R^2}u_r(s,t) = 0 \tag{41}$$

This differential equation can be integrated

$$_{rb}u_r(s,t) = u_2 \sin\left(\frac{s}{R}\right) + u_3 \cos\left(\frac{s}{R}\right) \tag{42}$$

Finally, replacing Eqs. (39) and (42) in Eq. (38b)

$$u_1 + u_2\,\frac{1}{R}\cos\left(\frac{s}{R}\right) - u_3\,\frac{1}{R}\sin\left(\frac{s}{R}\right) - \frac{u_\theta(s)}{R} = 0 \tag{43}$$

Therefore, the rigid body displacement fields of the curved beam is finally completed as

$$_{rb}u_\theta(s) = R\,u_1 + u_2\cos\left(\frac{s}{R}\right) - u_3\sin\left(\frac{s}{R}\right) \tag{44}$$

Equations (39), (42) and (44) represent the subspace of displacement fields of all rigid body displacements of the thick curved beam; u_1, u_2 and u_3 are generalized coordinates to represent rigid body displacements in the proposed thick curved-beam FE.

To complete the kinematic model of the proposed FE, quadratic deformation fields are assumed for $\varepsilon(s,t)$ and $\gamma(s,t)$ and linear curvature $\kappa(s,t)$:

$$\varepsilon(s,t) = \frac{\partial u_\theta(s,t)}{\partial s} + \frac{u_r(s,t)}{R} = \frac{1}{L}u_4 + \frac{s}{L^2}u_5 + \frac{s^2}{L^3}u_6 \tag{45a}$$

$$\gamma(s,t) = \alpha(s,t) + \frac{\partial u_r(s,t)}{\partial s} - \frac{u_\theta(s,t)}{R} = \frac{1}{L}u_7 + \frac{s}{L^2}u_8 + \frac{s^2}{L^3}u_9 \tag{45b}$$

$$\kappa(s,t) = \frac{\partial \alpha(s,t)}{\partial s} = \frac{1}{L^2}u_{10} + \frac{s}{L^3}u_{11} \tag{45c}$$

Therefore

$$_{def}\alpha(s,t) = \frac{s}{L^2}u_{10} + \frac{s^2}{2L^3}u_{11} \tag{46}$$

To solve for the other two displacement fields, let's define

$$_{def}u_r(s,t) = e_o + e_1 s + e_2 s^2 \tag{47a}$$

$$_{def}u_\theta(s,t) = g_0 + g_1 s + g_2 s^2 \tag{47b}$$

Imposing Eqs. (45a) and (45b),

$$\frac{\partial u_\theta(s,t)}{\partial s} + \frac{u_r(s,t)}{R} = g_1 + 2g_2 s + \frac{1}{R}e_o + \frac{1}{R}e_1 s + \frac{1}{R}e_2 s^2 = \frac{1}{L}u_4 + \frac{s}{L^2}u_5 + \frac{s^2}{L^3}u_6 \tag{48a}$$

$$\frac{\partial u_r(s,t)}{\partial s} - \frac{u_\theta(s,t)}{R} = e_1 + 2e_2 s - \frac{1}{R}(g_0 + g_1 s + g_2 s^2) = \frac{1}{L}u_7 + \frac{s}{L^2}u_8 + \frac{s^2}{L^3}u_9 - \frac{s}{L^2}u_{10} - \frac{s^2}{2L^3}u_{11} \tag{48b}$$

Equating terms of s of the same power in these equations

$$\begin{bmatrix} 1/R & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/R & 0 & 0 & 0 & 2 \\ 0 & 0 & 1/R & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/R & 0 & 0 \\ 0 & 0 & 2 & 0 & -1/R & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/R \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ g_0 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{L}u_4 \\ \frac{u_5}{L^2} \\ \frac{u_6}{L^3} \\ \frac{1}{L}u_7 \\ \frac{1}{L^2}u_8 - \frac{1}{L^2}u_{10} \\ \frac{1}{L^3}u_9 - \frac{1}{2L^3}u_{11} \end{bmatrix} \tag{49}$$

Solving for the interpolation-function coefficients:

$$g_2 = -\frac{R}{L^3}u_9 + \frac{R}{2L^3}u_{11} \tag{50a}$$

$$e_1 = \frac{Ru_5}{L^2} + \frac{2R^2}{L^3}u_9 - \frac{R^2}{L^3}u_{11} \tag{50b}$$

$$e_2 = \frac{R}{L^3}u_6 \tag{50c}$$

$$g_0 = -\frac{R}{L}u_7 + \frac{R^2u_5}{L^2} + \frac{2R^3}{L^3}u_9 - \frac{R^3}{L^3}u_{11} \quad (50d)$$

$$g_1 = -\frac{R}{L^2}u_8 + \frac{R}{L^2}u_{10} + 2\frac{R^2}{L^3}u_6 \quad (50e)$$

$$e_0 = \frac{R}{L}u_4 + \frac{R^2}{L^2}u_8 - \frac{R^2}{L^2}u_{10} - 2\frac{R^3}{L^3}u_6 \quad (50f)$$

Finally then

$${}_{def}u_r(s,t) = \frac{R}{L}u_4 + \frac{R^2}{L^2}u_8 - \frac{R^2}{L^2}u_{10} - 2\frac{R^3}{L^3}u_6 + (\frac{Ru_5}{L^2} + \frac{2R^2}{L^3}u_9 - \frac{R^2}{L^3}u_{11})s + \frac{R}{L^3}u_6 s^2 \quad (51a)$$

$${}_{def}u_\theta(s,t) = -\frac{R}{L}u_7 + \frac{R^2u_5}{L^2} + \frac{2R^3}{L^3}u_9 - \frac{R^3}{L^3}u_{11} + (-\frac{R}{L^2}u_8 + \frac{R}{L^2}u_{10} + 2\frac{R^2}{L^3}u_6)s + (-\frac{R}{L^3}u_9 + \frac{R}{2L^3}u_{11})s^2 \quad (51b)$$

Rearranging terms and considering rigid body and deformation components in the assumed displacement fields we finally obtain:

$$u_r(s,t) = 0u_1 + \sin\left(\frac{s}{R}\right)u_2 + \cos\left(\frac{s}{R}\right)u_3 + \frac{R}{L}u_4 + \frac{Rs}{L^2}u_5 + \left(-2\frac{R^3}{L^3} + \frac{R}{L^3}s^2\right)u_6 + 0u_7 + \frac{R^2}{L^2}u_8 + \frac{2sR^2}{L^3}u_9 - \frac{R^2}{L^2}u_{10} - \frac{sR^2}{L^3}u_{11} \quad (52a)$$

$$u_\theta(s,t) = R\,u_1 + \cos\left(\frac{s}{R}\right)u_2 - \sin\left(\frac{s}{R}\right)u_3 + 0u_4 + \frac{R^2}{L^2}u_5 + 2\frac{R^2s}{L^3}u_6 - \frac{R}{L}u_7 - \frac{Rs}{L^2}u_8 + (\frac{2R^3}{L^3} - \frac{Rs^2}{L^3})u_9 + \frac{Rs}{L^2}u_{10} + (-\frac{R^3}{L^3} + \frac{Rs^2}{2L^3})u_{11} \quad (52b)$$

$$\alpha(s,t) = u_1 + \frac{s}{L^2}u_{10} + \frac{s^2}{2L^3}u_{11} \quad (52c)$$

The proposed FE model has eleven generalized independent coordinates for each finite element with which any rigid body displacement can be represented avoiding shear locking and quadratic elongation and shear deformation fields, and linear curvature fields are captured. As in the case of the thin curved beam, the proposed FE formulation requires the inclusion of kinematic constraint equations to force displacement compatibility of boundary nodes of adjacent elements of the full structural model.

3.2 Potential and kinetic energy representation

The potential energy of the thick curved beam is expressed as

$$U = \frac{1}{2}\int_0^l (EA\varepsilon^2 + EI{\kappa_s}^2 + k_s GA\gamma^2)ds \quad (53)$$

where E is the Young modulus, A is the cross sectional area, I is the second moment of inertia of the cross sectional area, G is the shear modulus, and k_s is the cross section shear factor.

Substituting the expressions of longitudinal strain and curvature

$$U_e = \int_0^l (EA(\frac{\partial u_\theta(s,t)}{\partial s} + \frac{u_r(s,t)}{R})^2 + EI(\frac{\partial\alpha(s,t)}{\partial s})^2 + k_s GA(\alpha(s,t) + \frac{\partial u_r(s,t)}{\partial s} - \frac{u_\theta(s,t)}{R})^2)ds \quad (54)$$

Including rotational inertia of the thick curved beam, the kinetic energy can be expressed as

$$T = \frac{1}{2}\int_0^l (\rho A\frac{\partial u_r}{\partial t}^2 + \rho A\frac{\partial u_\theta}{\partial t}^2 + \rho I\frac{\partial\alpha}{\partial t}^2)ds \quad (55)$$

Replacing the displacement fields by the linear combination of the interpolating functions times the FE generalized coordinates given in Eqs. (52), the stiffness and mass matrices can be computed. The elements of the stiffness and mass matrices can be expressed in its general form expressed in Eqs. (23a) and (23b) where ${}_e\mathbf{K}$ and ${}_e\mathbf{M}$ are eleven by eleven FE matrices and the potential and kinetic energy expression are given in Eqs. (54) and (55). Closed form expressions are computed for the elements of the stiffness and mass matrices (which are not included for brevity).

3.3 Dynamic model and kinematic constraints

Given a structural model developed with the proposed FE as that shown in Figure 4, the equations of motion of the model in free vibration can be expressed as indicated in Eqs.(24) and (25) where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices of the model in generalized coordinates defined in column vector $\boldsymbol{r}$. $\boldsymbol{h}(\boldsymbol{r})$ is the vector of kinematic constraints associated to the imposed boundary conditions of the structure and the kinematic compatibility conditions imposed by continuity of radial and tangential displacement fields between adjacent elements.

Because the model considers eleven generalized coordinates for each FE and none of the generalized displacements in the radial or the tangential direction or the rotation are shared between adjacent elements in the proposed formulation, $\mathbf{M}$ and $\mathbf{K}$ are block diagonal matrices of size $11N_e \times 11N_e$ as indicated in Eqs. (26) and (27), where ${}_e^j\mathbf{K}$ and ${}_e^j\mathbf{M}$ are the FE stiffness and mass matrices of the thick curved FE.

To illustrate the assembly of the restraint matrix $\boldsymbol{L}$, let us consider a model of a thick curved beam clamped in one end and free in the other, with two finite elements of the type proposed (see Figure 4).

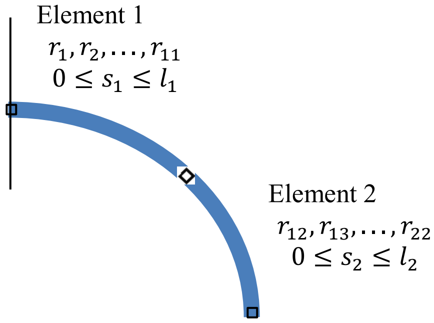

Figure 4. Two-element model of thin curved beam with proposed FE

The first three constraint equations on the generalized coordinates $\boldsymbol{r}$ are the radial displacement field, tangential displacement field and section rotation in element 1 equal to zero at $s_1 = 0$ (clamped end). The fourth to sixth constraints are the continuity of radial displacement, tangential displacement and rotation in the intersection of the elements of the model.

Defining as $r_1, r_2, \dots r_{11}$ the generalized coordinates associated to FE deformation coordinates $u_1, u_2, u_3, \dots, u_{11}$ of the first FE and $\mathrm{r}_{12}, \mathrm{r}_{13}, \dots \mathrm{r}_{22}$ to the deformation coordinates $u_1, u_2, u_3, \dots, u_{11}$ of the second FE, the six constraint equations can be expressed in the form of Eq. (25) with the kinematic constraint matrix $\boldsymbol{L}$ assembled with the constraints imposed by the clamped end:

$$h_1(\boldsymbol{r}) = {}_1u_r(0,t) = 0 \tag{56a}$$

$$h_2(\boldsymbol{r}) = {}_1u_\theta(0,t) = 0 \tag{56b}$$

$$h_3(\boldsymbol{r}) = {}_1\alpha(0,t) = 0 \tag{56c}$$

and the constraints imposed by displacement fields compatibility between adjacent elements 1 and 2:

$$h_4(\boldsymbol{r}) = \ {}_2u_r(0,t) - \ {}_1u_r(l_1,t) = 0 \tag{57a}$$

$$h_5(\boldsymbol{r}) = \ {}_2u_\theta(0,t) - \ {}_1u_\theta(l_1,t) = 0 \tag{57b}$$

$$h_6(\boldsymbol{r}) = \ {}_2\varphi(0,t) - \ {}_1\varphi(l_1,t) = 0 \tag{57c}$$

As developed in the case of a thin curved beam, the dynamic analysis can be done reducing the generalized coordinates to an independent set of coordinates $\boldsymbol{q}$, with a number of degrees of freedom $N_q = N_r - N_h$. In the example considered, $N_q = 22 - 6 = 16$ dofs.

Selecting $\boldsymbol{q}$ as an unconstraint subset of generalized coordinates $\boldsymbol{r}$, we can express the displacement coordinate vector of the complete model $\boldsymbol{r}(t) = \boldsymbol{L_{rq}}\,\boldsymbol{q}(t)$, solving for the slave displacements as linear functions of the unconstrained generalized coordinates $\boldsymbol{q}$, using Eq. (25). The unconstrained reduced order model is finally expressed as $\mathbf{M}_{\boldsymbol{q}}\ddot{\boldsymbol{q}}(t) + \mathbf{K}_{\boldsymbol{q}}\boldsymbol{q}(t) = \mathbf{0}$, where $\mathbf{M}_{\boldsymbol{q}}$ and $\mathbf{K}_{\boldsymbol{q}}$ are the reduced-order model mass and stiffness matrices.

If the model including thick and/or thin beam elements had non conservative external loads $\boldsymbol{F_r}$ assembled by virtual work on the generalized coordinates $\boldsymbol{r}$, the reduced-order model subjected to external loading take the form:

$$\mathbf{M}_{\boldsymbol{q}}\ddot{\boldsymbol{q}}(t) + \mathbf{K}_{\boldsymbol{q}}\boldsymbol{q}(t) = \boldsymbol{L_{rq}}^T\boldsymbol{F_r}(t) \tag{58}$$

4 NUMERICAL EVALUATION OF THE PROPOSED FE

In this section natural frequencies computed using the proposed FE are compared with results published in the literature for clamped-clamped beams for thin and thick curved beams.

4.1 Thin curved beam

A clamped-clamped curved beam is analyzed. The geometric parameters assumed in the numerical examples are defined as functions of the radius of gyration (Escanes et al., 2006),

$$r_g = \sqrt{I/A} \tag{59}$$

and the ratio of r_g/S where S is the total length of curved beam. Calling R to the radius of curvature of the thin beam and θ_o the total angle of the curved beam. $S = R\theta_o$. The number of elements used in the FE model is defined ad N_e and is varied in the analysis to assess the relative accuracy and convergence of the FE model for different meshes.

Once the finite element model is assembled for a given number N_e and a given ratio S/r_g, natural frequencies are calculated and normalized as

$$\lambda_j = \omega_j S^2 \sqrt{\frac{\rho A}{EI}} \tag{60}$$

where $\omega_j$is the j-th natural frequency computed with the proposed FE model.

Tables 1 and 2 show computed normalized fundamental natural frequency of the model as a function of $\mathrm{N_e}$ and $\mathrm{S/r_g}$ for the thin curved beam with rotational inertia included in the mass matrix and with only translational kinetic energy considered in the mass matrix. Table 3 compares the normalized fundamental frequency computed using the proposed FE with the corresponding value estimated by different authors. The results indicate good agreement.

	$N_e = 2$	$N_e = 4$	$N_e = 8$	$N_e = 12$	$N_e = 20$	$N_e = 30$
$S/r_g = 25$	37.8060	37.6932	37.6921	37.6921	37.6921	37.6921
$S/r_g = 50$	54.9906	54.9204	54.9101	54.9100	54.9100	54.9100
$S/r_g = 100$	55.6834	55.6109	55.6004	55.6003	55.6002	55.6002
$S/r_g = 150$	55.8090	55.7362	55.7257	55.7255	55.7255	55.7255
$S/r_g = 250$	55.8731	55.8001	55.7896	55.7894	55.7894	55.7894
$S/r_g = 350$	55.8907	55.8177	55.8071	55.8070	55.8070	55.8070
$S/r_g = 500$	55.9000	55.8270	55.8164	55.8163	55.8163	55.8163

Table 1. Normalized fundamental frequency λ_1 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thin beam model, translational and rotational inertia included in FE mass matrix.

	$N_e = 2$	$N_e = 4$	$N_e = 8$	$N_e = 12$	$N_e = 20$	$N_e = 30$
$S/r_g = 25$	38.1387	38.0315	38.0304	38.0304	38.0304	38.0304
$S/r_g = 50$	55.3619	55.2957	55.2857	55.2855	55.2855	55.2855
$S/r_g = 100$	55.7811	55.7097	55.6992	55.6991	55.6991	55.6991
$S/r_g = 150$	55.8529	55.7805	55.7700	55.7699	55.7698	55.7698
$S/r_g = 250$	55.8889	55.8161	55.8056	55.8054	55.8054	55.8054
$S/r_g = 350$	55.8988	55.8259	55.8153	55.8152	55.8151	55.8151
$S/r_g = 500$	55.9040	55.8310	55.8205	55.8203	55.8203	55.8203

Table 2. Normalized fundamental frequency λ_1 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thin beam model, only translational inertia included in FE mass matrix.

	Austin and Veletsos 1973	Kang et al. 1996	Escanes et al., 2006 $n_p = 13$	Proposed FE with $N_e = 12$ only translational inertia	Proposed FE with $N_e = 12$ translational and rotational inertia
$S/r_g = 25$	37.81	37.815	37.813	38.0304	37.6921
$S/r_g = 50$	54.98	54.973	54.985	55.2855	54.9100
$S/r_g = 100$	55.63	55.615	55.626	55.6991	55.6002
$S/r_g = 150$	55.74	55.732	55.742	55.7699	55.7255
$S/r_g = 250$	55.80	55.776	55.801	55.8054	55.7894
$S/r_g = 350$	55.83	55.812	55.817	55.8152	55.8070
$S/r_g = 500$	55.84	55.812	55.826	55.8203	55.8163

Table 3. Comparison of normalized fundamental frequency λ_1 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thin beam model with other published results.

	$N_e = 2$	$N_e = 4$	$N_e = 8$	$N_e = 12$	$N_e = 20$	$N_e = 30$
$S/r_g = 25$	37.1304	37.0571	36.9997	36.9965	36.9958	36.9957
$S/r_g = 50$	66.4179	52.8340	52.4213	52.3975	52.3923	52.3917
$S/r_g = 100$	72.8457	55.4130	54.9431	54.9160	54.9101	54.9094
$S/r_g = 150$	73.9106	55.9295	55.4477	55.4199	55.4138	55.4131
$S/r_g = 250$	74.4707	56.1991	55.7111	55.6828	55.6767	55.6760
$S/r_g = 350$	74.6268	56.2740	55.7842	55.7559	55.7498	55.7490
$S/r_g = 500$	74.7101	56.3139	55.8232	55.7948	55.7887	55.7880

Table 4. Normalized fundamental frequency λ_1 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thick beam model, translational and rotational inertia included in FE mass matrix.

Table 4 shows the normalized fundamental frequency computed using the proposed thick FE model for different number of elements and S/r_g ratios that correspond to relatively thin curved beams. Rotational inertia is included in the FE mass matrix and a Poisson ratio $\nu = 0.25$ is assumed. The curved beam has a total angle $\theta_o = \frac{\pi}{2}$ and is clamped in both ends.

4.2 Thick curved beam

Tables 5 through 8 show the normalized first four natural frequencies computed using the proposed thick FE model for different number of elements and S/r_g ratios that correspond to relatively thick curved beams. Rotational inertia is included in the FE mass matrix and a Poisson ratio $\nu = 0.25$ is assumed. As the table show very accurate estimations of the first four natural frequencies are obtained with a mesh of only 5 FE for the 90 degree angle arch.

	$N_e = 5$	$N_e = 10$	$N_e = 20$	$N_e = 30$	$N_e = 50$	$N_e = 100$
$S/r_g = 5$	11.0094	11.0084	11.0084	11.0084	11.0084	11.0084
$S/r_g = 10$	19.1928	19.1887	19.1884	19.1884	19.1884	19.1884
$S/r_g = 20$	31.3988	31.3843	31.3834	31.3833	31.3833	31.3833
$S/r_g = 50$	52.5806	52.4038	52.3923	52.3917	52.3916	52.3916
$S/r_g = 100$	55.1246	54.9232	54.9101	54.9094	54.9092	54.9092

Table 5. Normalized fundamental frequency λ_1 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thick beam model, translational and rotational inertia included in FE mass matrix.

	$N_e = 5$	$N_e = 10$	$N_e = 20$	$N_e = 30$	$N_e = 50$	$N_e = 100$
$S/r_g = 5$	13.4665	13.4641	13.4639	13.4639	13.4639	13.4639
$S/r_g = 10$	25.3912	25.3731	25.3719	25.3719	25.3718	25.3718
$S/r_g = 20$	40.7566	40.6727	40.6673	40.6670	40.6669	40.6669
$S/r_g = 50$	64.1521	63.9702	63.9585	63.9579	63.9577	63.9577
$S/r_g = 100$	96.4707	95.4147	95.3447	95.3409	95.3401	95.3400

Table 6. Normalized frequency λ_2 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thick beam model, translational and rotational inertia included in FE mass matrix.

	$N_e = 5$	$N_e = 10$	$N_e = 20$	$N_e = 30$	$N_e = 50$	$N_e = 100$
$S/r_g = 5$	18.6621	18.6572	18.6569	18.6568	18.6568	18.6568
$S/r_g = 10$	37.3382	37.3157	37.3141	37.3140	37.3140	37.3140
$S/r_g = 20$	69.4581	69.3883	69.3836	69.3833	69.3833	69.3833
$S/r_g = 50$	113.8951	112.8641	112.7873	112.7831	112.7822	112.7820
$S/r_g = 100$	156.0528	154.9826	154.9026	154.8981	154.8972	154.8970

Table 7. Normalized frequency λ_3 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thick beam model, translational and rotational inertia included in FE mass matrix.

	$N_e = 5$	$N_e = 10$	$N_e = 20$	$N_e = 30$	$N_e = 50$	$N_e = 100$
$S/r_g = 5$	25.8164	25.8096	25.8091	25.8091	25.8091	25.8091
$S/r_g = 10$	48.7992	48.6583	48.6485	48.6480	48.6479	48.6477
$S/r_g = 20$	79.6021	79.1027	79.0665	79.0645	79.0641	79.0640
$S/r_g = 50$	156.2830	154.3022	154.1394	154.1305	154.1285	154.1282
$S/r_g = 100$	188.4696	184.6264	184.2057	184.1823	184.1773	184.1765

Table 8. Normalized frequency λ_4 estimated with the proposed FE for $\theta_o = \frac{\pi}{2}$, thick beam model, translational and rotational inertia included in FE mass matrix.

To illustrate the slow convergence to natural frequencies and estimation error of a FE model built using straight cubic beam elements with shear deformation, the computed first two natural frequencies obtained using classical beam FE in structural models with different number of elements are shown in Tables 9 and 10.

	$N_e = 5$	$N_e = 10$	$N_e = 20$	$N_e = 30$	$N_e = 50$	$N_e = 100$
$S/r_g = 5$	16.4284	16.3339	16.3055	16.3000	16.2972	16.2960
$S/r_g = 10$	24.6080	24.1562	24.0347	24.0118	24.0000	23.9950
$S/r_g = 20$	33.6074	33.0873	32.9457	32.9189	32.9051	32.8993
$S/r_g = 50$	57.8647	56.0771	55.4944	55.3793	55.3195	55.2940
$S/r_g = 100$	58.3765	56.5158	55.9142	55.7956	55.7340	55.7078

Table 9. Normalized frequency λ_1 estimated with a straight beam FE discretization mesh for $\theta_o = \frac{\pi}{2}$, shear deformation included in FE stiffness matrix; translational and rotational inertia included in FE mass matrix.

	$N_e = 5$	$N_e = 10$	$N_e = 20$	$N_e = 30$	$N_e = 50$	$N_e = 100$
$S/r_g = 5$	20.1914	19.6808	19.5542	19.5308	19.5189	19.5138
$S/r_g = 10$	31.1344	30.9049	30.8404	30.8281	30.8217	30.8190
$S/r_g = 20$	51.7327	50.7006	50.3562	50.2876	50.2519	50.2367
$S/r_g = 50$	66.5739	65.3546	65.0326	64.9716	64.9401	64.9268
$S/r_g = 100$	103.3130	99.0592	97.6949	97.4187	97.2741	97.2123

Table 10. Normalized frequency λ_2 estimated with a straight beam FE discretization mesh for $\theta_o = \frac{\pi}{2}$, shear deformation included in FE stiffness matrix; translational and rotational inertia included in FE mass matrix.

5 CONCLUSIONS

The proposed FE for thin and thick curved beam or circular arch shows a satisfactory performance in terms of accuracy, convergence and mesh size. Natural frequency estimates of a 90 degree clamped-clamped arch have been analyzed and compared with estimations published in scientific literature. With meshes of only five finite elements of the proposed type excellent accuracy is obtained in the estimation of the first four natural frequencies of the analyzed structure. The FE doesn't show shear locking because it can represent in the exact rigid body displacement fields for the linear kinematic theory. The accuracy obtained using relatively small meshes (large elements compared to the number required using straight beam 2-node elements) is directly related to the capability of representing quadratic elongation and shear deformation in the FE space variable

The need of handling slave and independent generalized displacement coordinates for structural models that use the proposed FE of thin or thick curved beams could be considered a limitation in the application of the FE to standard finite-element codes which are frequently developed using FEs with nodal displacements as generalized coordinates, that guarantee adjacent element displacement compatibility without the need of imposing constraints as those of the type defined in Eq. (25). In any case, handling linear constraints and order reduction to independent generalized coordinates in linear MK models is not a complex or a numerically expensive task. This means that the proposed FE can be implemented along with standard node-based FE including the corresponding linear constraint definition, coordinate selection and order reduction methods as necessary steps to complete a standard linear model for modal analysis or direct integration of equations of motion of structures including thin and/or thick curved beams.

The extension of the proposed approach to variable geometric parameter models of curved beams is a line of future research.

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