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A STUDY ON THE EFFECT OF TWO INTERNAL TRANSLATIONAL ELASTIC RESTRAINTS ON MODE SHAPES OF BEAMS

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Abstract. This work deals with the problem of free vibrations of uniform beams with elastically restrained ends and with two internal translational elastic restraints. The main objective of this work is to obtain the minimum stiffness of the internal elastic restraints that raises a natural frequency to its upper limit. The minimum stiffness is determined by using the derivative of the function which gives the natural frequencies, with respect to the support position. The problem is solved with the close form solution. The effect of mode shape shift caused by changes in the rigidity parameter of the internal translational elastic restraints is analyzed. Additionally, results of the frequency parameter and modal shape of beams with different end conditions are presented.

1 INTRODUCTION

The vibration of Euler–Bernoulli beams with elastic restrictions have been extensively studied. It is not possible to give a detailed account by reason of the great amount of information; nevertheless, some relevant references will be cited. Particularly, several investigators have studied the influence of elastic restraints at the ends of vibrating beams (Mabie and Rogers, 1968; Mabie and Rogers, 1972; Lee, 1973; Mabie and Rogers, 1974; Grant, 1975; Hibbeler, 1975; Maurizi et al., 1976; Goel, 1976a, b; Grossi and Laura, 1982; Laura and Grossi, 1982; Cortinez and Laura, 1985; Laura and Gutierrez, 1986; Grossi and Bhat, 1991; Grossi et al., 1993; Nallim and Grossi, 1999). Exact frequency and normal mode shape expressions have been derived for uniform beams with ends elastically restrained against rotation and translation (Rao and Mirza, 1989). Excellent handbooks have appeared in the literature giving frequencies, tables and mode shape expressions (Blevins, 1979; Karnovsky and Lebed, 2004).

The problem of vibrations of beams elastically restrained at intermediate points has also been extensively treated. Grossi and Albarracín (2003) determined the exact eigenfrequencies of a uniform beam with intermediate elastic restrictions.

The minimum stiffness of a point support that raises a natural frequency of a beam to its upper limit has been investigated by several researchers. Courant and Hilbert (1953) has demonstrated that the optimum location of a rigid support should be at the nodal points of a higher vibration mode. Akesson and Olhoff (1988) showed that in the case of elastic supports the optimum locations are the same as that of rigid supports and that there exists a minimum stiffness of an additional elastic support whenever the fundamental frequency of a uniform cantilever beam is increased to its maximum. Wang (2003) determined the minimum stiffness of an internal elastic support to maximize the fundamental frequency of a vibrating beam. Wang et al. (2006) derived the closed-form solution for the minimum stiffness of a simple point support that raises a natural frequency of a beam to its upper limit. Raffo and Grossi (2011, 2012) studied the effects on natural frequencies and mode shapes of beams with an intermediate elastic support obtaining the exact value of its rigidity when a modal shift occurs. Finally Raffo and Grossi (2014) studied the effect of an internal elastic translational restriction on mode shape of beams with internal hinges.

The above review of the literature reveals that many efforts had been devoted to the analysis of the influence of elastic restraints parameters, located at the ends and at intermediate points, on the vibrating characteristics of beams. However, the influence on frequencies and mode shapes of varying two intermediate supports located at nodal points of higher modes has not been studied. There is no paper that presents a complete analysis of the mentioned effects of two intermediate elastic supports in a beam generally restrained at both ends.

The aim of the present paper is to investigate the natural frequencies and mode shapes of a beam with two arbitrarily located internal translational elastic restrictions and ends elastically restrained against rotation and translation. Adopting the adequate values of the rotational and translational restraints parameters at the ends, all the possible combinations of classical end conditions, (i.e.: clamped, simply supported, sliding and free) can be generated. The existence of a critical value of the dimensionless restraint parameters of the two internal translational restraints which determines the interchange of roles of the corresponding modal shapes of two consecutive non-dimensional frequency parameters is demonstrated. The later allows to obtain the minimum value of the restraint parameters of the two internal translational restraints that maximizes a natural frequency.

The classical method of separation of variables has been used for the determination of the exact frequencies and mode shapes. The algorithm developed can be applied to a wide range of elastic restraint conditions.

Tables and figures are given for frequencies, and two-dimensional plots for mode shapes are included. A great number of problems were solved and, since the number of cases is prohibitively large, results are presented for only a few cases.

2 THE BOUNDARY VALUE PROBLEM

Let us consider a beam of length l, which has elastically restrained ends and has two intermediate translational elastic restrictions, as shown in Figure 1. The beam system is made up of three different spans, which correspond to the intervals $\left[0,c_1\right],\left[c_1,c_2\right]$ and $\left[c_2,l\right]$ respectively. The rotational restraints located at the ends of the beam are characterized by the parameters $r_L,r_R,$ and the translational restraints by $t_L,t_R,t_{c_i},i=1,2.$ Adopting the adequate values of the parameters r_L,r_R and t_L,t_R all the possible combinations of classical end conditions can be generated. By using $t_{c_i},i=1,2,$ the effects of the internal hinges and intermediate restraints are taken into account.

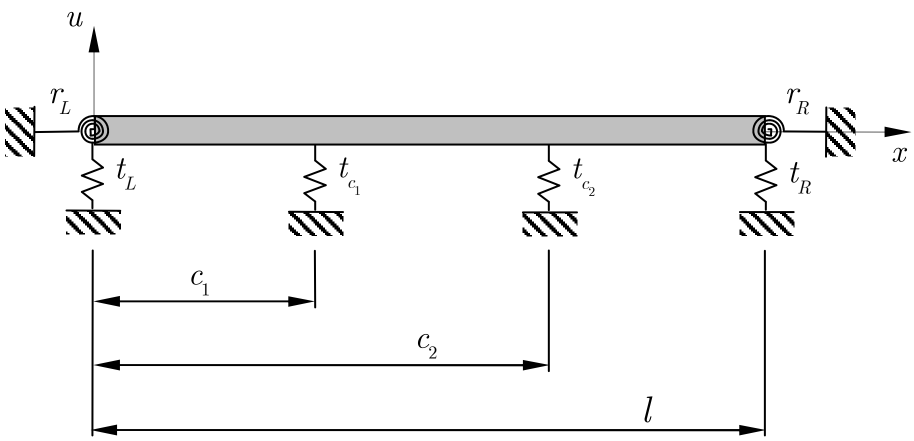


Figure 1: Mechanical system under study.

In order to analyze the transverse planar displacements of the system under study, we suppose that the vertical position of the beam at any time t is described by the function $u=u\left(x,t\right),\ x\in\left[0,l\right]$. It is well known that at time t the kinetic energy of the beam can be expressed as

$$T_b=\frac{1}{2}\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left(\rho A\right)_i\left(x\right)\left(\frac{\partial u}{\partial t}\left(x,t\right)\right)^2dx \tag{1}$$

where $\left(\rho A\right)_i=\rho_iA_i$ denotes the mass per unit length of the $i-$th span and $c_0=0,c_3=l.$

The total potential energy due to the elastic deformation of the beam, the elastic restraints at the ends and the intermediate elastic restraints, is given by:

$$U=\frac{1}{2}\left\{\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left(EI\right)_i\left(x\right)\left(\frac{\partial^2 u}{\partial x^2}\left(x,t\right)\right)^2 dx+r_L\left(\frac{\partial u}{\partial x}\left(0^+,t\right)\right)^2\right.$$
$$\left.+r_R\left(\frac{\partial u}{\partial x}\left(l^-,t\right)\right)^2+\sum_{i=0}^{3}t_{c_i}u^2\left(c_i,t\right)\right\}, \tag{2}$$

where $\left(EI\right)_i=E_iI_i$ denotes the flexural rigidity of the $i-$th span, $t_{c_0}=t_L$, $t_{c_3}=t_R$ and the notations 0^+ and l^- imply the use of lateral limits and lateral derivatives.

Hamilton´s principle requires that between times t_a and t_b, at which the positions are known, the motion will make stationary the action integral $F(u)=\int_{t_a}^{t_b}Ldt$ on the space of admissible functions, where the Lagrangian L is given by $L=T_b-U$. In consequence, the energy functional to be considered is given by

$$F\left(u\right)=\frac{1}{2}\int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left(\left(\rho A\right)_i\left(x\right)\left(\frac{\partial u}{\partial t}\left(x,t\right)\right)^2-\left(EI\right)_i\left(x\right)\left(\frac{\partial^2 u}{\partial x^2}\left(x,t\right)\right)^2\right)dx\right]dt-$$
$$-\frac{1}{2}\int_{t_a}^{t_b}\left[r_L\left(\frac{\partial u}{\partial x}\left(0^+,t\right)\right)^2+r_R\left(\frac{\partial u}{\partial x}\left(l^-,t\right)\right)^2+\sum_{i=0}^{3}t_{c_i}u^2\left(c_i,t\right)\right]dt. \tag{3}$$

The stationary condition for the functional given by Eq. (3) requires that

$$\delta F\left(u;v\right)=0,\forall v\in D_a, \tag{4}$$

where $\delta F\left(u,v\right)$ is the first variation of F at u in the direction v and D_a is the space of admissible directions at u for the space D of admissible functions. In order to make the mathematical developments required by the application of the techniques of the calculus of variations, we assume that $\left(\rho A\right)_i\in C\left(\left[c_{i-1},c_i\right]\right)$, $\left(EI\right)_i\in C^2\left(\left[c_{i-1},c_i\right]\right)$, $i=1,2,3$.

The space D is the set of functions $u\left(x,\bullet\right)\in C^2\left[t_a,t_b\right]$, $u\left(\bullet,t\right)\in C^1\left(\left[0,l\right]\right)$, $u\left(\bullet,t\right)\Big|_{\left[c_{i-1},c_i\right]}\in C^4\left(\left[c_{i-1},c_i\right]\right)$, $i=1,2,3$.

In view of all these observations and since Hamilton´s principle requires that at times t_a and t_b the positions are known, the space D is given by

$$D=\left\{u;u\left(x,\bullet\right)\in C^2\left[t_a,t_b\right],u\left(\bullet,t\right)\in C^1\left(\left[0,l\right]\right),u\left(\bullet,t\right)\Big|_{\left[c_{i-1},c_i\right]}\in C^4\left(\left[c_{i-1},c_i\right]\right),\right.$$
$$\left.i=1,2,3,u\left(x,t_a\right),u\left(x,t_b\right)\text{ prescribed},\forall x\in\left[0,l\right]\right\}. \tag{5}$$

The only admissible directions v at $u\in D$ are those for which $u+\varepsilon v\in D$ for sufficiently small ε and $\delta F(u;v)$ exists. In consequence, and in view of Eq. (5), v is an admissible direction at u for D if, and only if, $v \in D_a$ where

$$D_a = \Big\{ v; v(x,\bullet) \in C^2[t_a, t_b], v(\bullet,t) \in C^1([0,l]), v(\bullet,t)\big|_{[c_{i-1},c_i]} \in C^4([c_{i-1}, c_i]), \\ i = 1,2,3, v(x,t_a) = v(x,t_b) = 0, \forall x \in [0,l] \Big\}. \tag{6}$$

The development of the techniques of Calculus of Variations (Grossi, 2010) allows obtaining that the function u must satisfy the differential equations:

$$\frac{\partial^2}{\partial x^2}\left((EI)_i(x) \frac{\partial^2 u}{\partial x^2}(x,t) \right) + (\rho A)_i(x) \frac{\partial^2 u}{\partial t^2}(x,t) = 0, \\ \forall x \in (c_{i-1}, c_i), i = 1,2,3, t \geq 0, \tag{7}$$

with the following boundary and transitions conditions:

$$r_L \frac{\partial u}{\partial x}(0^+,t) = (EI)_1 \frac{\partial^2 u}{\partial x^2}(0^+,t), \tag{8}$$

$$t_L u(0^+,t) = -\frac{\partial}{\partial x}\left((EI)_1 \frac{\partial^2 u}{\partial x^2}(0^+,t) \right), \tag{9}$$

$$w(c_i^-,t) = w(c_i^+,t), i = 1,2, \tag{10}$$

$$\frac{\partial u}{\partial x}(c_i^+,t) = \frac{\partial u}{\partial x}(c_i^-,t), i = 1,2, \tag{11}$$

$$(EI)_i \frac{\partial^2 u}{\partial x^2}(c_i^-,t) = (EI)_{i+1} \frac{\partial^2 u}{\partial x^2}(c_i^+,t), i = 1,2, \tag{12}$$

$$t_{c_i} u(c_i,t) = \frac{\partial}{\partial x}\left((EI)_i \frac{\partial^2 u}{\partial x^2}(c_i^-,t) \right) - \frac{\partial}{\partial x}\left((EI)_{i+1} \frac{\partial^2 u}{\partial x^2}(c_i^+,t) \right), i = 1,2, \tag{13}$$

$$r_R \frac{\partial u}{\partial x}(l^-,t) = -(EI)_3 \frac{\partial^2 u}{\partial x^2}(l^-,t), \tag{14}$$

$$t_R u(l^-,t) = \frac{\partial}{\partial x}\left((EI)_3 \frac{\partial^3 u}{\partial x^3}(l^-,t) \right), \tag{15}$$

where $t \geq 0$.

Eqs. (8), (9), (14) and (15) correspond to the boundary conditions and Eqs. (10) to (13) correspond to the transition conditions.

3 NATURAL FREQUENCIES AND MODE SHAPES

Using the well-known method of separation of variables, when the mass per unit length and the flexural rigidity at the spans are the same, we assume as solutions of Eqs. (7) the functions given by the series

$$u_i\left(x,t\right) = \sum_{n=1}^{\infty} u_{i,n}\left(x\right)\cos\omega t,\; i = 1,2,3, \tag{16}$$

where $u_{i,n}$ are the corresponding nth modes of natural vibration. If we consider the change of variable $\overline{x} = x / l$ into Eqs. (7) and (8)-(15), the close form solution of the mechanical system under study $u_{i,n}$ is given by

$$u_{1,n}\left(\overline{x}\right) = A_1\cosh\lambda\overline{x} + A_2\sinh\lambda\overline{x} + A_3\cos\lambda\overline{x} + A_4\sin\lambda\overline{x}, \quad \forall\overline{x}\in\left[o,\overline{c}_1\right], \tag{17}$$

$$u_{2,n}\left(\overline{x}\right) = A_5\cosh\lambda\overline{x} + A_6\sinh\lambda\overline{x} + A_7\cos\lambda\overline{x} + A_8\sin\lambda\overline{x}, \quad \forall\overline{x}\in\left[\overline{c}_1,\overline{c}_2\right], \tag{18}$$

$$u_{3,n}\left(\overline{x}\right) = A_9\cosh\lambda\overline{x} + A_{10}\sinh\lambda\overline{x} + A_{11}\cos\lambda\overline{x} + A_{12}\sin\lambda\overline{x}, \quad \forall\overline{x}\in\left[\overline{c}_2,1\right], \tag{19}$$

where $\overline{c}_i = c_i / l$ and

$$\lambda^4 = \frac{\rho A}{EI}\omega^2 l^4. \tag{20}$$

Substituting Eqs. (17)-(19) into Eq. (16) and then in the boundary conditions given by Eqs. (8), (9), (14), (15) and transition conditions defined by Eqs. (10) to (13), expressed in the new variable $\overline{x}$, we obtain a set of twelve homogeneous equations in the constants A_i. Since the system is homogeneous in order to obtain a non-trivial solution the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G\left(T_L, R_L, T_R, R_R, T_{c_i}, \lambda, \overline{c}_i\right) = 0,\; i = 1,2, \tag{21}$$

where

$$T_L = \frac{t_L l^3}{EI},\; R_L = \frac{r_L l}{EI},\; T_R = \frac{t_R l^3}{EI},\; R_R = \frac{r_R l}{EI},\; T_{c_i} = \frac{t_{c_i} l^3}{EI},\; i = 1,2. \tag{22}$$

The values of the frequency parameter $\lambda = \left(\left(\rho A / EI\right)\omega^2\right)^{1/4} l,$ were obtained with the classical bisection method.

4 NUMERICAL RESULTS

In order to describe the corresponding boundary conditions the symbolism SS identifies a simply supported end, C a clamped end, F a free end and ER identifies an elastically restrained end. The ER-ER beam analyzed corresponds to a beam with $T_L = R_L = 10$ and $T_R = R_R = 1$. Since the number of cases which can be analyzed by the developed algorithm is prohibitively large, results are presented only for a few cases.

Table 1 presents the values of λ_1 and λ_2 with their corresponding modal shapes and the nodal positions of mode 2 that is used to define the values of $\bar{c}_2$ when $T_{c_2} = 0$, for SS-SS, C-F and ER-ER beams with different values of T_{c_1}, with $\bar{c}_1 = 1/3$ for the SS-SS and C-F beams and $\bar{c}_1 = 0.3$ for the ER-ER beam.

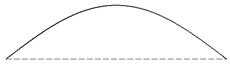
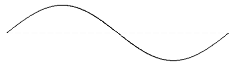
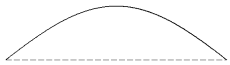
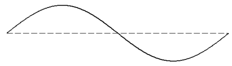
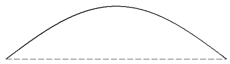
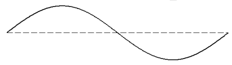
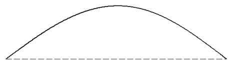
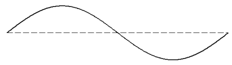
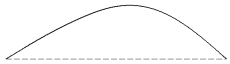
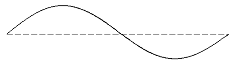
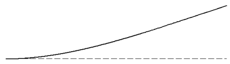
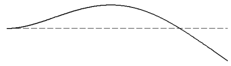
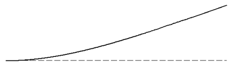
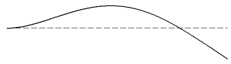
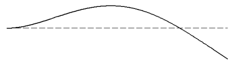
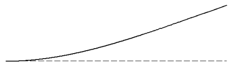
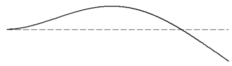
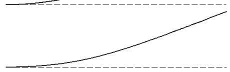
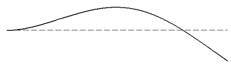
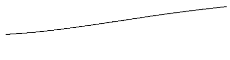
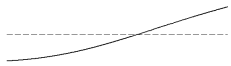
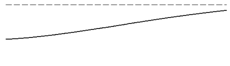
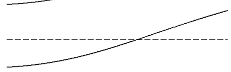
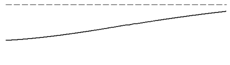
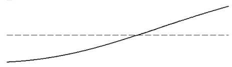
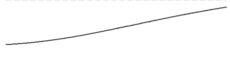
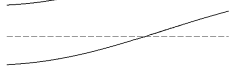
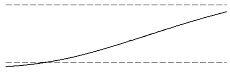
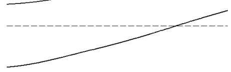
B. C.	T_{c_1}	λ_1	λ_2	$\bar{c}_2$	Modal shape 1	Modal shape 2
SS-SS	0	3.141593	6.283185	0.500000		
	0.1	3.142801	6.283336	0.500017		
	1	3.153604	6.284698	0.500168		
	10	3.254898	6.298387	0.501683		
	100	3.898462	6.441740	0.516723		
C-F	0	1.875104	4.694091	0.783445		
	0.1	1.875519	4.694427	0.783463		
	1	1.879232	4.697447	0.783633		
	10	1.914009	4.727195	0.785278		
	100	2.125682	4.984839	0.798026		
ER-ER	0	1.684567	2.827619	0.589373		
	0.1	1.688026	2.828749	0.589753		
	1	1.717695	2.838966	0.593159		
	10	1.918884	2.944736	0.625149		
	100	2.255251	3.790038	0.760771		

Table 1: Values of λ_1 and λ_2 with their corresponding modal shapes and the nodal positions of mode 2 that define the values of $\bar{c}_2$ for SS-SS, C-F and ER-ER ($T_L = R_L = 10$, $T_R = R_R = 1$) beams with different values of T_{c_1}, with $\bar{c}_1 = 1/3$ for the SS-SS and C-F beams and $\bar{c}_1 = 0.3$ for the ER-ER beam.

Based on the concepts presented, a numerical procedure has been developed with the purpose of determining the critical value $T^{(1,2)}$ of T_{c_2}, such that over it the values of λ_1 cannot be raised further whereas the values of the coefficient λ_2 increases (Raffo and Grossi, 2014).

Based on the cases presented in Table 1, Tables 2 to 14 show the results obtained when $0 < T_{c_2} < \infty$, analyzing the minimum stiffness of each case denoted with $T_{c_2} = T^{(1,2)}$ and the modal shape effect when $T_{c_2} = \left(1 \pm 0.05\right) T^{(1,2)}$.

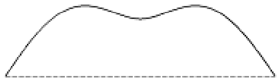
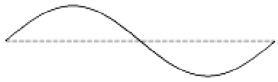
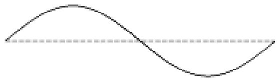
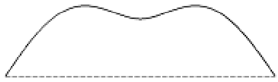
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	6.230371	6.283185
		
$T^{(1,2)} = 995.913544$		
$1.05\ T^{(1,2)}$	6.283185	6.332961
		

Table 2: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\bar{c}_1 = 1/3$, $\bar{c}_2 = 0.5$, $T_{c_1} = 0$ and different values of T_{c_2}.

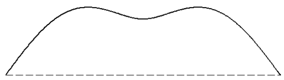
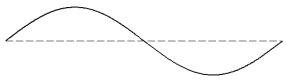
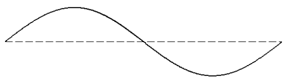
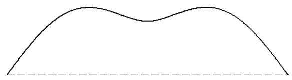
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	6.230526	6.283336
		
$T^{(1,2)} = 995.912695$		
$1.05\ T^{(1,2)}$	6.283336	6.333108
		

Table 3: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\bar{c}_1 = 1/3$, $\bar{c}_2 = 0.500017$, $T_{c_1} = 0.1$ and different values of T_{c_2}.

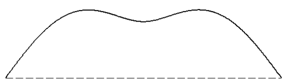
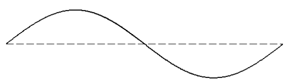
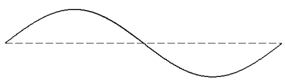
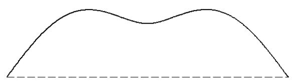
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	6.231926	6.284698
		
$T^{(1,2)} = 995.905986$		
$1.05\ T^{(1,2)}$	6.284698	6.334434
		

Table 4: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\bar{c}_1 = 1/3$, $\bar{c}_2 = 0.500168$, $T_{c_1} = 1$ and different values of T_{c_2}.

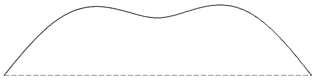
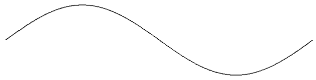
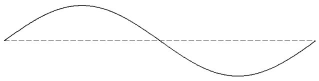
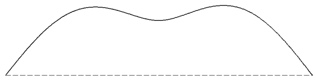
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	6.245986	6.298387
		
$T^{(1,2)} = 995.930720$		
$1.05\ T^{(1,2)}$	6.298387	6.347783
		

Table 5: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\bar{c}_1 = 1/3$, $\bar{c}_2 = 0.501683$, $T_{c_1} = 10$ and different values of T_{c_2}.

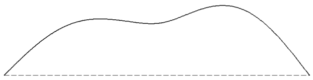
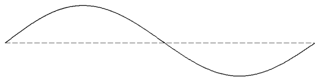
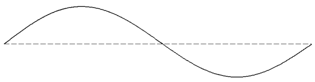
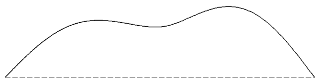
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	6.392114	6.441740
		
$T^{(1,2)} = 1005.167312$		
$1.05\ T^{(1,2)}$	6.441740	6.488572
		

Table 6: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\bar{c}_1 = 1/3$, $\bar{c}_2 = 0.516723$, $T_{c_1} = 100$ and different values of T_{c_2}.

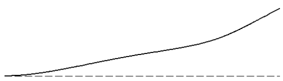
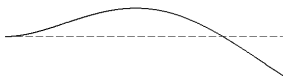
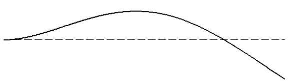
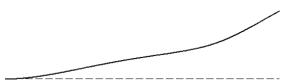
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	4.642480	4.694427
		
$T^{(1,2)} = 266.918888$		
$1.05\ T^{(1,2)}$	4.694427	4.744104
		

Table 7: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\bar{c}_1 = 1/3$, $\bar{c}_2 = 0.783463$, $T_{c_1} = 0.1$ and different values of T_{c_2}.

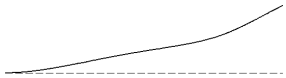
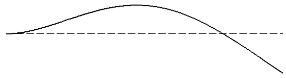
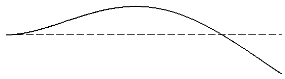
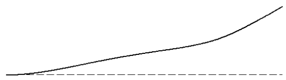
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	4.645474	4.697447
		
$T^{(1,2)} = 267.353932$		
$1.05\ T^{(1,2)}$	4.697447	4.747150
		

Table 8: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\overline{c}_1 = 1/3$, $\overline{c}_2 = 0.783633$, $T_{c_1} = 1$ and different values of T_{c_2}.

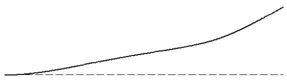
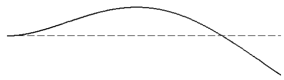
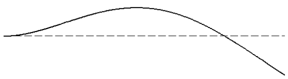
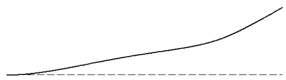
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	4.674955	4.727195
		
$T^{(1,2)} = 271.691003$		
$1.05\ T^{(1,2)}$	4.727195	4.777155
		

Table 9: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\overline{c}_1 = 1/3$, $\overline{c}_2 = 0.785278$, $T_{c_1} = 10$ and different values of T_{c_2}.

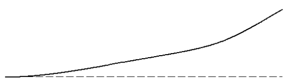
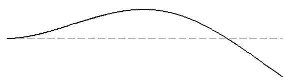
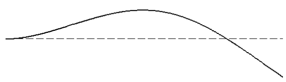
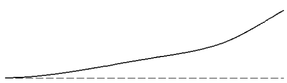
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	4.930073	4.984839
		
$T^{(1,2)} = 313.240970$		
$1.05\ T^{(1,2)}$	4.984839	5.037222
		

Table 10: Values λ_1 and λ_2 and their corresponding mode shapes of a SS-SS beam with $\overline{c}_1 = 1/3$, $\overline{c}_2 = 0.798026$, $T_{c_1} = 100$ and different values of T_{c_2}.

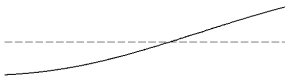
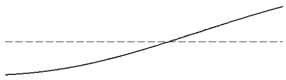
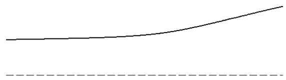
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	2.800542 	2.828749 
$T^{(1,2)} = 56.730087$		
$1.05\ T^{(1,2)}$	2.828749 	2.855839 

Table 11: Values λ_1 and λ_2 and their corresponding mode shapes of a ER-ER beam with $\overline{c}_1 = 0.3$, $\overline{c}_2 = 0.589753$, $T_{c_1} = 0.1$ and different values of T_{c_2}.

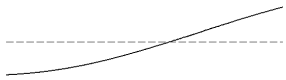
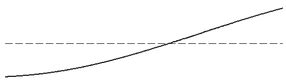
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	2.810874 	2.838966 
$T^{(1,2)} = 56.735622$		
$1.05\ T^{(1,2)}$	2.838966 	2.865954 

Table 12: Values λ_1 and λ_2 and their corresponding mode shapes of a ER-ER beam with $\overline{c}_1 = 0.3$, $\overline{c}_2 = 0.593159$, $T_{c_1} = 1$ and different values of T_{c_2}.

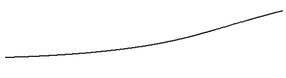
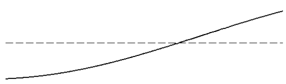
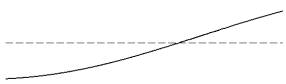
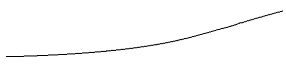
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	2.916960 	2.944736 
$T^{(1,2)} = 57.757581$		
$1.05\ T^{(1,2)}$	2.944736 	2.971475 

Table 13: Values λ_1 and λ_2 and their corresponding mode shapes of a ER-ER beam with $\overline{c}_1 = 0.3$, $\overline{c}_2 = 0.625149$, $T_{c_1} = 10$ and different values of T_{c_2}.

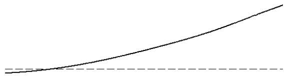
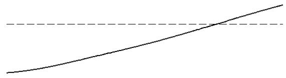
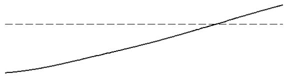
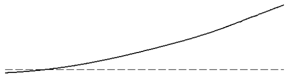
T_{c_2}	λ_1	λ_2
$0.95\ T^{(1,2)}$	3.749950 	3.790038 
$T^{(1,2)} = 100.549000$		
$1.05\ T^{(1,2)}$	3.790038 	3.828685 

Table 14: Values λ_1 and λ_2 and their corresponding mode shapes of a ER-ER beam with $\bar{c}_1 = 0.3$, $\bar{c}_2 = 0.760771$, $T_{c_1} = 100$ and different values of T_{c_2}.

Table 2 to 6 depicts the first two exact values of the frequency coefficient λ and the corresponding mode shapes of a SS-SS beam with $\bar{c}_1 = 1/3$, $T_{c_1} = 0,\ 0.1,\ 1,\ 10,\ 100$ respectively, for different values of $\bar{c}_2$ and $T_{c_2} = \left(1 \pm 0.05\right) T^{(1,2)}$.

Table 7 to 10 depicts the first two exact values of the frequency coefficient λ and the corresponding mode shapes of a C-F beam with $\bar{c}_1 = 1/3$, $T_{c_1} = 0.1,\ 1,\ 10,\ 100$ respectively, for different values of $\bar{c}_2$ and $T_{c_2} = \left(1 \pm 0.05\right) T^{(1,2)}$.

Table 11 to 14 depicts the first two exact values of the frequency coefficient λ and the corresponding mode shapes of a ER-ER beam with $T_L = R_L = 10$, $T_R = R_R = 1$, $\bar{c}_1 = 1/3$, $T_{c_1} = 0.1, 1, 10, 100$ respectively, for different values of $\bar{c}_2$ and $T_{c_2} = \left(1 \pm 0.05\right) T^{(1,2)}$.

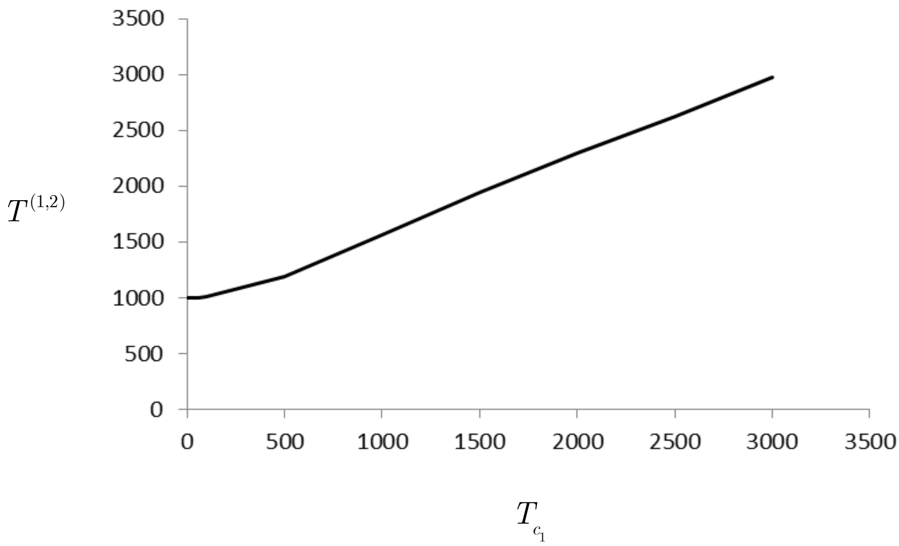


Figure 2: Values of $T^{(1,2)}$ as a function of T_{c_1} for a SS-SS beam with $\bar{c}_1 = 1/3$.

Figure 2 shows the values of $T^{(1,2)}$ obtained from different values of $T_{\bar{c}_1}$ and $\bar{c}_2$ of a SS-SS beam with $\bar{c}_1 = 1/3$.

5 CONCLUSIONS

The minimum exact value of the stiffness of an elastic translational restraint that raises the first natural frequency of a beam with the presence of a second elastic translational restraint to its upper limit was obtained.

The analyzed cases of a SS-SS, C-F, and ER-ER beams show that the behavior of the exact value of the intermediate elastic translational restriction and the behavior of the first natural frequency parameter agree with the presented by Raffo and Grossi (2011, 2012, 2014).

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