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DATUM DEFINITION IN THE SOLUTION R-MINOS OF THE ADJUSTMENT OF A FREE TRILATERATION NETWORK

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Abstract. Free trilateration networks in the 2D and 3D space, ranging from a local to a global scale, are continuously designed and established with a wide variety of objectives such as: cartographic, geodynamics, civil engineering, cadastral among others, using – for example - the Global Positioning System (GPS) among others Global Navigation Satellite Systems (GNSS). To evaluate the quality of the adjustment of a free trilateration network it is very useful to characterize properly the datum definition involved. The geodetic datum definition is the set of all conventions, algorithms and constants necessaries to define and realize the origin, orientation, scale and their time evolution of a Geodetic Reference System (GRS), in such a way that these attributes are accessible to the users through occupation, direct or indirect observation. In this work, we deal with the adjustment of a two-dimensional free trilateration network constituted by physical points**,** where distances between these points have been observed. For the network adjustment, it is used a coordinate based formulation in a no stochastic linear model through a underdetermined consistent system of indirect linear observational equations. The network point positions are defined in a Geodetic Reference System of Cartesian Coordinate (x,y): GRS(x,y). The GRS(x,y) is characterized by : a) right-handed convention is adopted for the axis; b) the origin is a point not specified of the Earth ; c) the ox (+) and oy(+) axis do not have specified orientations; d) the scale or length defined of the unit vectors along ox and oy is the meter (SI), and it is realized by the observed distances of the trilateration network. The lack of definition in the origin and orientation of the GRS(x,y) cause a datum defect and a rank-deficiency in the design matrix. The solution of the adjustment of the free trilateration network called Minimum Norm Solution with respect to the R–seminorm (R-MINOS) of the underdetermined consistent system of indirect linear observational equations is obtained based in an optimum criterion, which resolves the datum problem. The set of the physical points of the trilateration with coordinates given by the R-MINOS is the Geodetic Reference Frame of Cartesian Coordinate (x,y): GRF(x,y)-(R-MINOS). In this work, the GRF(x,y)-(R-MINOS) is characterized when R is the identity matrix (I) and when R is a positive semidefinite matrix. It is shown that, for R equal to I, the realization of the origin and orientation of the GRS(x,y) is given through the fulfillment of the conditions “No Net Translation” (NNT) and “No Net Rotation” (NNR) respectively. As a numerical example, the GRF(x,y)-(R-MINOS) is characterized in the adjustment of a free two-dimensional trilateration network with six points when R is the identity matrix (I) and when R is a positive semidefinite matrix.

I. INTRODUCTION

The geodetic datum definition is the set of all conventions, algorithms and constants necessaries to define and realize the origin, orientation, scale and their time evolution of a Geodetic Reference System (GRS), in such a way that these attributes be accessible to the users through occupation, direct or indirect observation. (Vacaflor, p.2647, 2010).

In this work, we deal with the adjustment of a free two-dimensional trilateration network constituted by " k " physical points P_i , where " n " distances $s_{ij}^{obs}, i<j, i=1...k, j=1...k$ between these points have been observed. For the network adjustment , it is used a coordinate based formulation in a no stochastic linear model through the underdetermined consistent system of indirect linear observational equations : $y_{n\times1}=A_{n\times m}x_{m\times1}$, also known as "the first problem of algebraic regression" (Grafarend, p.14, 2006) ; (Grafarend and Awange, p.23, 2012), $y \in R(A)$, $r(A):=q$, $q=n\langle m$, $n=$ number of observed distances; $m=$ number of unknown parameters $r=$rank, $m=2k$, $R(A)=$ column space of $A_{n\times m}$, $y_{n\times1}=$ vector of observations , $x_{m\times1}$ =vector of unknown parameters.

We were motivated to study the solution of the above mentioned adjustment problem known as R-MINOS (developed in the following sections) since, for R=I, the Geodetic Reference Frame of Cartesian Coordinate (x,y) with coordinates given by the solution I-MINOS designated as $GRF(x,y)_{I-MINOS}$, fulfill two important conditions : "No Net Translation" (NNT) and "No Net Rotation" (NNR) . Let us take into account here that the NNR condition is used to provide orientation time evolution in the International Terrestrial Reference Frame (ITRF) datum definition (Petit and Luzum, p.34, 2010). In this sense, investigations on the stability of a geodetic no-net-rotation frame and its implication for the International Terrestrial Reference Frame were recently published (Kreemer et.al, 2006). Moreover, the NNT and NNR conditions are involved in the definition of an ideal terrestrial reference system (Kovalevsky and Mueller, p.7, 1989). On the other hand, unlike to a previous research (Vacaflor, 2010) where the NNT and NNR conditions were studied in a rank-deficient Singular Gauss-Markov Model (SGMM) – *stochastic* linear model -, know we consider here a *no stochastic* adjustment linear model to study these conditions.

The network point positions are defined in a Geodetic Reference System of Cartesian Coordinate (x,y): GRS(x,y).

The GRS(x,y) is characterized by : a) a right-handed axis system is adopted ; b) the origin **"o"** is a point **P** not specified of the Earth ; c) the first and second rays are the **ox** and **oy** positive axis respectively with not specified orientations; d) the unit of length is the meter (SI) , and it is realized by the observed distances of the trilateration network.

The lack of definition in the origin and orientation of the GRS(x,y) cause a datum defect and a rank-deficiency $m-q=3$ in the design matrix $A_{n\times m}$ with $d=:m-q=3$, $d=$ number of datum defect.

II. SOLUTION R-MINOS OF THE ADJUSTMENT OF A FREE TRILATERATION NETWORK

Let us consider a free two dimensional trilateration network constituted by " k " physical points P_i with coordinates (x_i, y_i), $i=1...k$ in the $GRS(x,y)$, and related through " n " observed distances, and *not* being *defined* for any epoch, the position and orientation of the $GRS(x,y)$.

Moreover, let us consider that the a priori coordinates (x_i^0, y_i^0), $i=1... k$ from the reference frame $GRF(x_0, y_0)$ are known.

For the network adjustment, it is used a coordinate based formulation in a no stochastic linear model through the underdetermined consistent system of indirect linear observational equations:

$$y_{nx1} = A_{nxm} x_{mx1} \ , \ y \in R(A), \ r(A) := q \ , \ q = n \langle m \qquad (1)$$

The lack of definition in the origin and orientation of the GRS(x,y) cause a datum defect and a rank-deficiency in (1).

with,

$n =$ number of observations; $m =$ number of unknown parameters.

$r =$ rank; $d =$ number of datum defect.

$y_{nx1} =$ vector of observations (increments).

$$y_{nx1} = [y_{ij}] = [(s_{12}^{obs} - s_{12}^0), (s_{13}^{obs} - s_{13}^0), ..., (s_{ij}^{obs} - s_{ij}^0), ..., (s_{k-1,k}^{obs} - s_{k-1,k}^0)]^T$$

$$s_{ij}^0 = \sqrt{(\Delta x_{ij}^0)^2 + (\Delta y_{ij}^0)^2} \ , \ i = 1...k \ , \ j = 1...k \ , \ i < j$$

$$\Delta x_{ij}^0 = x_j^0 - x_i^0 \ ; \ \Delta y_{ij}^0 = y_j^0 - y_i^0$$

$A_{nxm} =$ Design or coefficient matrix (“Jacobian”).

$$A_{nxm} = \begin{bmatrix} \alpha_{12} \\ ... \\ \alpha_{ij} \\ ... \\ \alpha_{k-1,k} \end{bmatrix} \ ; \ \alpha_{ij_{1xm}} = [0, ..., -\Delta x_{ij}^0, -\Delta y_{ij}^0, ..., \Delta x_{ij}^0, \Delta y_{ij}^0, ..., 0].(1/s_{ij}^0)$$

$R(A) =$ column space of A_{nxm}

$x_{mx1} =$ Vector of unknown parameters (coordinate increments).

$x_{mx1} = X_{mx1} - X_{mx1}^0$

$X_{mx1} =$ Vector of unknown coordinates of the points P_i of the GRF(x,y) expressed in the $GRS(x, y)$.

$$X_{mx1} = [x_1, y_1 ... \ x_k, y_k]^T$$

$X_{mx1}^0 =$ Vector of known coordinates of P_i of the “a priori” or “approximated” $GRF(x_0, y_0)$.

$$X_{mx1}^0 = [x_1^0, y_1^0 ... \ x_k^0, y_k^0]^T$$

$$x_{mx1} = [dx_1 \quad dy_1 \quad ... \quad dx_k \quad dy_k]^T \ ; \ dx_i = x_i - x_i^0 \ ; \ dy_i = y_i - y_i^0, \ i = 1...k \ , m = 2k$$

The solution to the adjustment problem in model (1) known as $x_{M_{mx1}}$ R-MINOS (Minimum Norm Solution with respect to the R–seminorm) is obtained based in the optimum criterion $\|x_M\|_R^2 := x_M^T R x_M \leq x^T R x =: \|x\|_R^2$, which resolves the datum problem, with $x \in \Re^m$ representing all other vectors solution of $y_{nx1} = A_{nxm} x_{mx1}$. In this sense, the following definitions and theorems are given here without demonstration (Grafarend and Schaffrin,p.11,1993).

Definition 1:

$x_{M_{mx1}}$ is R-MINOS (Minimum Norm Solution with respect to the **R** –seminorm) of:

$$y_{nx1} = A_{nxm} x_{mx1} \ , \ y \in R(A), \ r(A) := q \tag{2}$$

When:

$$y_{nx1} = A_{nxm} x_{M_{mx1}} \tag{3}$$

And with respect to the all other vectors solution "x" of $y_{nx1} = A_{nxm} x_{mx1}$, $x \in \Re^m$:

$$\|x_M\|_R^2 := x_M^T R x_{M \le} x^T R x =: \|x\|_R^2 \tag{4}$$

Theorem 1:

$x_{M_{mx1}}$ is R-MINOS if:

$$\begin{bmatrix} R & A^T \\ A & 0 \end{bmatrix} . \begin{bmatrix} x_M \\ \lambda_M \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} \tag{5}$$

With the vector $\lambda_{M_{mx1}}$ of Lagrange multipliers, $x_{M_{mx1}}$ always exists and it is unique if:

$$r\left[R \quad , A^T\right] = m \tag{6}$$

Or equivalently, $R + A^T A$ is regular.

Theorem 2 :

$x_{M_{mx1}} = L_{mxn} y_{nx1}$ is R-MINOS of (2) $\forall \ y \in R(A)$ when the matrix L_{mxn} satisfies the following two conditions:

$$ALA = A \text{ or } L = A^- ; \ RLA = (RLA)^T \tag{7}$$

A^- :=generalized inverse of A.

In this case, $Rx_M = RLy$ is always unique.

Hence, using the Definition 1 in the model Eq. (1) for:

$$R_{mxm} = I_{mxm}, \ r(A) := q, \ q = n \langle m \tag{8}$$

I_{mxm} =Identity matrix

We get the following:

Definition 2:

$x_{M_{mx1}}$ is I-MINOS (Minimum Norm Solution with respect to the **I** –seminorm) of Eq.(1):

$$y_{nx1} = A_{nxm} x_{mx1} \ , \ y \in R(A) \ , \ r(A) := q \ , \ q = n \langle m$$

When Eq.(3):

$$y_{nx1} = A_{nxm} x_{M_{mx1}}$$

And with respect to the all other vectors solution "x" of $y_{nx1} = A_{nxm} x_{mx1}$, $x \in \Re^m$:

$$\|x_M\|_I^2 := x_M^T I x_M \leq x^T I x =: \|x\|_I^2 \tag{9}$$

In this case, exist a right inverse of A :

$$A_{RI}^- := A^T (AA^T)^{-1} \tag{10}$$

With the property: $AA_{RI}^- = I_n$ (11)

Therefore, from the Theorem 2 with Eq. (8) and Eq. (12):

$$L := A_{RI}^- \tag{12}$$

It is obtained a unique vector solution $x_{M_{mx1}}$ I-MINOS (Minimum Norm Solution with respect to the **I** –seminorm) of Eq. (1) as:

$$x_{M_{mx1}} = Ly = A_{RI}^- y = A^T \left(AA^T\right)^{-1} y \tag{13}$$

With Eq. (9), $\|x_M\|_I^2 := x_M^T I x_{M \leq} x^T I x =: \|x\|_I^2$

When R is a positive semidefinite matrix, the following condition are fulfilled $r[R \quad , A^T] = m$ or equivalently $R + A^T A$ is regular.

The general solution (Grafarend, E.W. y Schaffrin, B, 1993, p.10) $x_{M_{mx1}}$ is R-MINOS (Minimum Norm Solution with respect to the **R** –seminorm):

$$x_{M_{mx1}} = (R + A^T A)^{-1} A^T [A(R + A^T A)^{-1} A^T]^- y \tag{14}$$

The Eq. (14) is independent of the g-inverse $[A(R + A^T A)^{-1} A^T]^-$, hence, $x_{M_{mx1}}$, R-MINOS, can be obtained as:

$$x_{M_{mx1}} = (R + A^T A)^{-1} A^T [A(R + A^T A)^{-1} A^T]^{-1} y \tag{15}$$

III. THE GEODETIC REFERENCE FRAME OF R-MINOS. CHARACTERIZATION OF THE ORIGIN AND ORIENTATION REALIZED OF THE GRS(x,y)

III.1. To characterize the origin and orientation realized of the GRS(x,y) by the Geodetic Reference Frame of Cartesian Coordinate (x,y) with coordinates given by the $x_{M_{mx1}}$ I-MINOS designated as $GRF(x, y)_{I-MINOS}$, the following methodology is presented:

1) A matrix E_{3xm} which spans the null space of A is introduced with the condition:

$$AE^T = 0 \; , \; o(E) = dxm = 3xm \; , \; r(E) = d = 3 \tag{16}$$

$$\Rightarrow \qquad R(A^T) \overset{\perp}{\oplus} R(E^T) = \Re^m \tag{17}$$

For a free two-dimensional trilateration network, E_{3xm} is (Vacaflor, 2010):

$$E_{3xm}=\begin{bmatrix}1 & 0 & \dots & 1 & 0\\ 0 & 1 & \dots & 0 & 1\\ -y_1^0 & x_1^0 & \dots & -y_k^0 & x_k^0\end{bmatrix} \tag{18}$$

2) It is perfomed and analized the product Ex_M as follows:

$E_{3xm}x_{M\,mx1}=EA^T(AA^T)^{-1}y=0_{3x1}$, since $EA^T=0$, due to Eq.(16)

Hence,

$$E_{3xm}x_{M_{mx1}}=0_{3x1} \tag{19}$$

If E_{3xm} is partitioned as:

$$E_{3xm}:=\begin{bmatrix}E1_{2xm}\\ E2_{1xm}\end{bmatrix} \tag{20}$$

with:

$$E1_{2xm}:=\begin{bmatrix}1 & 0 & \dots & 1 & 0\\ 0 & 1 & \dots & 0 & 1\end{bmatrix} \tag{21}$$

$$E2_{1xm}:=[-y_1^0 \quad x_1^0 \quad -y_2^0 \quad x_2^0 \quad \dots \quad -y_k^0 \quad x_k^0] \tag{22}$$

Substituting E_{3xm} of Eq. (20) in Eq. (19) leads to:

$$\begin{bmatrix}E1_{2xm}\\ E2_{1xm}\end{bmatrix}x_{M_{mx1}}=\begin{bmatrix}0_{2x1}\\ 0_{1x1}\end{bmatrix} \tag{23}$$

$$\Rightarrow \qquad E1_{2xm}.x_{M_{mx1}}=0_{2x1} \quad \text{(NNT)} \tag{24}$$

$$\Rightarrow \qquad E2_{1xm}\,x_{M_{mx1}}=0_{1x1} \quad \text{(NNR)} \tag{25}$$

Hence, for R=I the $GRF(x,y)_{I-MINOS}$ of the ***I****-MINOS* (Minimum Norm Solution with respect to the **I** –seminorm, I:= identity matrix) $x_{M_{mx1}}=A^T\left(AA^T\right)^{-1}y$ fulfill the conditions : “No Net Translation” (NNT) since $E1_{2xm}.x_{M_{mx1}}=0_{2x1}$ and “No Net Rotation” (NNR) since $E2_{1xm}\,x_{M_{mx1}}=0_{1x1}$ respectively (Vacaflor, p.2652, 2010). Hence, the origin and orientation realized of the GRS(x,y) by the $GRF(x,y)_{I-MINOS}$ is given through the fulfillment of the conditions:

(i) The centre of gravity $C_g\left(x_{I-MINOS},y_{I-MINOS}\right)$ of the network as defined by the adjusted coordinates of the $GRF(x,y)_{I-MINOS}$ is equal to the centre of gravity $C_g\left(x_0,y_0\right)$ of the network as defined by the approximate coordinates of the $GRF(x_0,y_0)$, $C_g\left(x_{I-MINOS},y_{I-MINOS}\right)=C_g\left(x_0,y_0\right)$.

(ii) The average orientation of the network as defined by the approximate coordinates of the $GRF(x_0,y_0)$ is maintained.

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III.2. To characterize the origin and orientation realized of the GRS(x,y) by the Geodetic Reference Frame of Cartesian Coordinate (x,y) with coordinates given by the $x_{M_{m\times 1}}$ R-MINOS designated as $GRF(x,y)_{R-MINOS}$ when R is a positive semidefinite matrix , the following methodology is presented:
it is considered that:

$$R_{m\times m} := diag(1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad ... \quad 0 \quad 0 \quad 0) \tag{26}$$

and $x_{M_{m\times 1}}$,R-MINOS is obtained as $x_{M_{m\times 1}} = (R + A^T A)^{-1} A^T [A(R + A^T A)^{-1} A^T]^{-1} y$ according Eq. (15).

Using Eq. (26) and Eq.(15) it is obtained the coordinates of the Geodetic Reference Frame of R-MINOS: $GRF(x,y)_{R-MINOS}$:

$$x_{M_{m\times 1}} = [0 \quad 0 \quad 0 \quad dy_{M_2} \quad dx_{M_3} \quad dy_{M_3} \quad dx_{M_4} \quad dy_{M_4} \quad dx_{M_5} \quad dy_{M_5} \quad dx_{M_6} \quad dy_{M_6}] \tag{27}$$

Hence, the origin and orientation realized of the GRS(x,y) by the $GRF(x,y)_{R-MINOS}$ when $R_{m\times m} := diag(1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad ... \quad 0 \quad 0 \quad 0)$ is given through the fulfillment of the conditions:

$$\begin{aligned} &\text{a) } dx_{M_1} = x_{M_1} - x_1^0 = 0 \Rightarrow x_{M_1} = x_1^0 \\ &\text{b) } dy_{M_1} = y_{M_1} - y_1^0 = 0 \Rightarrow y_{M_1} = y_1^0 \\ &\text{c) } dx_{M_2} = x_{M_2} - x_2^0 = 0 \Rightarrow x_{M_2} = x_2^0 \end{aligned} \tag{28}$$

Since the $GRF(x,y)_{I-MINOS}$ fulfill the NNT and NNR conditions, then, we consider the I-MINOS solution more advantageous than the R-MINOS - R according to (26) - for the two-epoch geodetic deformation analysis using free trilateration networks.

IV. EXAMPLE. ADJUSTMENT OF A FREE TRILATERATION NETWORK (k=6, n=9). THE GRF(x,y) OF R-MINOS. CHARACTERIZATION OF THE ORIGIN AND ORIENTATION REALIZED OF THE GRS(x,y).

Let us consider a free two-dimensional trilateration constituted by "$k=6$" physical points P_i with coordinates (x_i, y_i), $i = 1... 6$ in the $GRS(x,y)$, and the points related through "n=9" observed distances s_{ij}^{obs}(between P_i and P_j, see Table 1), and *not* being *defined* for any epoch, the position and orientation of the $GRS(x,y)$.

Moreover, let us consider that the a priori coordinates (x_i^0, y_i^0), $i = 1... 6$ of the reference frame $GRF(x_0, y_0)$ are available (see Table 2).

s_{ij}^{obs}	Distance (m)	s_{ij}^{obs}	Distance (m)
s_{12}	3899.4269	s_{34}	3298.1854
s_{15}	3805.8666	s_{45}	2813.9446
s_{16}	4891.0000	s_{46}	4203.1897
s_{23}	3107.9294	s_{56}	4113.1502
s_{25}	2079.9391		

Table 1: Observed distances in a two-dimensional trilateration with six points.

Pto.	$x^0(m)$	$y^0(m)$
1	0	0
2	275	3900
3	2250	6300
4	4800	4200
5	2200	3100
6	4900	0

Table 2: Coordinates of P_i of a two-dimensional trilateration from the $GRF(x_0, y_0)$.

To perform the adjustment it is used a coordinate based formulation in a no stochastic linear model through the underdetermined consistent system of indirect linear observational equations from Eq. (1), with $n = 9$, and $m = 12$:

$$y_{9x1} = A_{9x12} x_{12x1} \;,\;\; y \in R(A) \;,\; r(A) := q = 9,\; q = n \langle m \qquad (29)$$

$r =$ rank; $d =: m - q = 3$ = number of datum defect.

y_{9x1} =vector of observations (increments)

$$y_{9x1} = [y_{ij}] = [(s_{12}^{obs} - s_{12}^0), (s_{15}^{obs} - s_{15}^0), ..., (s_{ij}^{obs} - s_{ij}^0), ..., (s_{56}^{obs} - s_{56}^0)]^T$$

$$s_{ij}^0 = \sqrt{(\Delta x_{ij}^0)^2 + (\Delta y_{ij}^0)^2} \;,$$

$$\Delta x_{ij}^0 = x_j^0 - x_i^0 \;\; ; \;\; \Delta y_{ij}^0 = y_j^0 - y_i^0$$

A_{9x12} =Design or coefficient matrix (“Jacobian”)

$$A_{9x12} = \begin{bmatrix} \alpha_{12} \\ ... \\ \alpha_{ij} \\ ... \\ \alpha_{46} \end{bmatrix} ; \;\; \alpha_{ij_{1x12}} = [0, ..., -\Delta x_{ij}^0, -\Delta y_{ij}^0, ..., \; \Delta x_{ij}^0, \Delta y_{ij}^0, ..., \; 0].(1/s_{ij}^0)$$

x_{12x1} =Vector of unknown parameters (coordinate increments).

$$x_{12x1} = X_{12x1} - X_{12x1}^0$$

X_{12x1} = Vector of unknown coordinates of the points P_i of the GRF(x,y) expressed in the $GRS(x,y)$.

$X_{12x1} = [x_1, y_1 ... x_6, y_6]^T$

X^0_{12x1} = Vector of known coordinates of P_i of the “a priori” or “approximated” $GRF(x_0, y_0)$.

$X^0_{12x1} = [x^0_1, y^0_1 ... x^0_6, y^0_6]^T$

$x_{12x1} = [dx_1 \quad dy_1 \quad ... \quad dx_6 \quad dy_6]^T$; $dx_i = x_i - x^0_i$; $dy_i = y_i - y^0_i$, $i = 1...6$, $m = 2k = 12$

The $x_{M_{12x1}}$ I-MINOS (Minimum Norm Solution with respect to the **I** –seminorm) of Eq. (29) is according to Eq. (13):

$$x_{M_{12x1}} = Ly = A^{-}_{RI} y = A^T \left(AA^T\right)^{-1} y \tag{30}$$

$$x_{M_{12x1}} = \begin{bmatrix} 3.5537 \\ 3.4669 \\ 1.9333 \\ -6.7009 \\ -0.6023 \\ -4.9059 \\ -2.8837 \\ 0.5382 \\ 3.4452 \\ 9.1246 \\ -5.4463 \\ -1.5228 \end{bmatrix} (m)$$

Following the methodology presented in III.1 to characterize the origin and orientation realized of the GRS(x,y) by the Geodetic Reference Frame of Cartesian Coordinate (x,y) with coordinates given by the $x_{M_{mx1}}$ I-MINOS designated as $GRF(x,y)_{I-MINOS}$, it is observed that $x_{M_{12x1}} = A^T \left(AA^T\right)^{-1} y$ fulfill the conditions : “No Net Translation” (NNT) since:

$$E1_{2x12} x_{M_{12x1}} = 0_{2x1} \text{ (NNT)}, \; E2_{1x12} x_{M_{12x1}} = 0_{1x1} \text{ (NNR)} \tag{31}$$

With:

$$E1_{2x12} := \begin{bmatrix} 1 & 0 & ... & 1 & 0 \\ 0 & 1 & ... & 0 & 1 \end{bmatrix} \tag{32}$$

$$E2_{1x12} := [-y^0_1 \quad x^0_1 \quad -y^0_2 \quad x^0_2 \quad ... \quad -y^0_6 \quad x^0_6] \tag{33}$$

From (9): $\left\|x_{M_{12x1}}\right\|_I^2 := x_M^T I x_M = 233.4321m^2 \le x^T I x =: \left\|x_{12x1}\right\|_I^2$.

Hence, the origin and orientation realized of the GRS(x,y) by the $GRF(x,y)_{I-MINOS}$ is given through the fulfillment of the conditions:

(i) The centre of gravity $C_g\left(x_{I-MINOS}, y_{I-MINOS}\right)$ of the network as defined by the adjusted coordinates of the $GRF(x,y)_{I-MINOS}$ is equal to the centre of gravity $C_g\left(x_0, y_0\right)$ of the network as defined by the approximate coordinates of the $GRF(x_0,y_0)$, $C_g\left(x_{I-MINOS}, y_{I-MINOS}\right) = C_g\left(x_0, y_0\right)$.

(ii) Maintains the average orientation of the network as defined by the approximate coordinates of the $GRF(x_0,y_0)$.

To characterize the origin and orientation realized of the GRS(x,y) by the Geodetic Reference Frame of Cartesian Coordinate (x,y) given by the $x_{M_{mx1}}$ R-MINOS designated as $GRF(x,y)_{R-MINOS}$ when R is a positive semidefinite matrix , the following methodology is presented:

it is considered that:

$$R_{12x12} := diag(1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad 0) \qquad (34)$$

And according to Eq. (15) $x_{M_{12x1}}$,R-MINOS is obtained as:

$$x_{M_{12x1}} = (R + A^T A)^{-1} A^T [A(R + A^T A)^{-1} A^T]^{-1} y \qquad (35)$$

$$x_{M_{12x1}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -10.2821 \\ -1.5385 \\ -9.3077 \\ -4.6924 \\ -4.9230 \\ 1.1794 \\ 4.7436 \\ -9.0000 \\ -7.0255 \end{bmatrix} (m)$$

Hence, the origin and orientation realized of the GRS(x,y) by the $GRF(x,y)_{R-MINOS}$ when $R_{12x12} := diag(1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad 0)$, is given through the fulfillment of the conditions:

a) $dx_{M_1} = x_{M_1} - x_1^0 = 0 \Rightarrow x_{M_1} = x_1^0 = 0m$

b) $dy_{M_1} = y_{M_1} - y_1^0 = 0 \Rightarrow y_{M_1} = y_1^0 = 0m$ (36)

c) $dx_{M_2} = x_{M_2} - x_2^0 = 0 \Rightarrow x_{M_2} = x_2^0 = 275m$

From the applications point of view, the example shows that the solution R-MINOS can be easily implemented - when the conditions of model (29) are satisfied - in the adjustment of a two-dimensional free GPS vector networks, constituted by "k" physical points related through "n" observed baselines. This type of network ranging from a local to a global scale, are continuously designed and established with a wide variety of objectives such as: cartographic, geodynamics, civil engineering, cadastral among others.

V.CONCLUSION

In dealing with the adjustment of a two-dimensional free trilateration network constituted by the physical points $P_i, i = 1...k$, where distances between these points have been observed , it is shown the solution $x_{M_{m\times 1}}$,R-MINOS (Minimum Norm Solution with respect to the **R** – seminorm, R is a positive semidefinite matrix) using a coordinate based formulation in a non stochastic linear model through the underdetermined consistent system of indirect linear observational equations $y_{nx1} = A_{nxm} x_{mx1}$, $y \in R(A)$, $r(A) := q$, $q = n \langle m$, obtained based in the optimal criterion, which resolves the datum problem : $\|x_M\|_R^2 := x_M^T R x_M \leq x^T R x =: \|x\|_R^2$, $x \in \Re^m$, where "x" are all other vectors solution of $y_{nx1} = A_{nxm} x_{mx1}$. It is shown that, for R=I the coordinates of the $GRF(x,y)_{I-MINOS}$ given by the ***I**-MINOS* (Minimum Norm Solution with respect to the **I** –seminorm, I:= identity matrix) : $x_{M_{m\times 1}} = A^T\left(AA^T\right)^{-1} y$ fulfill the conditions : "No Net Translation" (NNT) since $E1_{2xm}.x_{M_{m\times 1}} = 0_{2x1}$and "No Net Rotation" (NNR) since $E2_{1xm}\, x_{M_{m\times 1}} = 0_{1x1}$ respectively . Hence, the origin and orientation realized of the GRS(x,y) by the $GRF(x,y)_{I-MINOS}$ is given through the fulfillment of the conditions : (i) The centre of gravity $C_g\left(x_{I-MINOS}, y_{I-MINOS}\right)$of the network as defined by the adjusted coordinates of the $GRF(x,y)_{I-MINOS}$ is equal to the centre of gravity $C_g\left(x_0, y_0\right)$ of the network as defined by the approximate coordinates of the $GRF(x_0,y_0)$, $C_g\left(x_{I-MINOS}, y_{I-MINOS}\right) = C_g\left(x_0, y_0\right)$; (ii) Maintains the average orientation of the network as defined by the approximate coordinates of the $GRF(x_0,y_0)$. It is also shown that, when $R_{mxm} := diag(1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ ...\ \ 0\ \ 0\ \ 0)$, $x_{M_{m\times 1}}$,R-MINOS can be obtained as $x_{M_{m\times 1}} = (R + A^T A)^{-1} A^T [A(R + A^T A)^{-1} A^T]^{-1} y$, where the origin and orientation realized of the GRS(x,y) by the $GRF(x,y)_{R-MINOS}$ is given through the fulfillment of the conditions: a) $dx_{M_1} = x_{M_1} - x_1^0 = 0 \Rightarrow x_{M_1} = x_1^0$;b) $dy_{M_1} = y_{M_1} - y_1^0 = 0 \Rightarrow y_{M_1} = y_1^0$ and
c) $dx_{M_2} = x_{M_2} - x_2^0 = 0 \Rightarrow x_{M_2} = x_2^0$.

VI. REFERENCES

- Grafarend, W.E. and Schaffrin, B. (1993): Ausgleichungsrechnung in linearen Modellen. Mannheim; Leipzig; Wien; Zürich: BI-Wiss.-Verl. ISBN: 3-411-16381-X, 483p.

- Grafarend, W.E. (2006): Linear and nonlinear models: fixed effects, random effects, and mixed models. Hubert & Co. GmbH & Co. Kg, Göttingen. Walter de Gruyter GmbH & Co. KG, 10785 Berlin. ISBN-13: 978-3-11-016216-5, 773p.

- Grafarend, W.E. and Awange, J.L. (2012): Applications of Linear and Nonlinear Models. Fixed Effects, Random Effects, and Total Least Squares. Springer. Heidelberg. New York. Dordrecht.London. ISBN 978-3-642-22240-5,1027p.

- Kovalevsky, J., Mueller, I.I. and Kolaczek, B. (1989): Reference Frames in Astronomy and Geophysics. Dordrecht, The Netherlands. Kluwer Academic Publishers. ISBN-13: 978-94-010-6909-0, 469p.

- Kreemer, C., Lavallée, D.A., Blewitt, G. and Holt, W.E. (2006): On the stability of a geodetic no-net-rotation frame and its implication for the International Terrestrial Reference Frame. Geophysical Research Letters, Vol. 33,pp.1-5.

- Petit, G. and Luzum, B. (2010): IERS Conventions (2010). IERS Technical Note No.36. Verlag des Bundesants fur katographie und Geodasie. Frankfurt and Maim. ISSN: 1019-4568.179p.

- Vacaflor, J. L. (2010): Geodetic datum and No Net Translation (NNT), No Net Rotation (NNR) conditions from transformation parameters, a reference frame and a selection matrix. Mecánica Computacional, Vol. XXIX, Eduardo N. Dvorkin, Marcela B. Golschmit y Mario A. Storti (Eds). Asociación Argentina de Mecánica Computacional. Editorial: Imprenta Macagno S.R.L. Santa Fe. ISSN: 1666-6070, pp. 2647-2653.