Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXVI, págs. 921-930 (artículo completo)
José G. Etse, Bibiana M. Luccioni, Martín A. Pucheta, Mario A. Storti (Eds.)
San Miguel de Tucumán, 6-9 Noviembre 2018

QUALITATIVE ANALYSIS OF DYNAMICAL PROBLEMS USING A COMBINATION OF SYMBOLIC ALGEBRA COUPLED WITH POINCARÉ-LINDSTEDT METHOD

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Keywords: Symbolic method, analytical approximations, Perturbation method.

Abstract. Dynamical problems are governed by initial value problems (IVP), often nonlinear, that must be solved to understand the dynamical features of a problem. Numerical methods are very efficient and provide approximations with the precision required, if one is interested in solving a specific problem. Unfortunately, numerical approximations do not provide the insight necessary to understand how a solution depends on the parameters of the problem. Sometimes, perturbation methods help in the sense that they can provide an analytical approximation that shows how the parameters influence the solutions. However to solve a problem for large time intervals require high order approximations that are cumbersome to derive. This paper uses a symbolic method to derive approximation of an IVP using Poincaré-Lindstedt method. The resulting linear problems are combined to have several orders of approximations. The approximations are compared to understand their domain of validity. As a reference to estimate the quality of an approximation, a Runge-Kutta method is used for a specific value of the parameters, of course. To show the main features of the methodology, it is applied to a non-damped Duffing equation, the simplest nonlinear problem used in Mechanics. It is computed analytical approximations of displacement, velocity, and frequency, for any order of approximation and initial conditions desired by the user. To quantify how the order of approximation affect the results, the obtained analytical approximations are evaluated for different combinations of parameters values. The results show that the number of terms has a great influence in the accuracy of the approximation, specially when the term that controls the nonlinearities grows.

1 INTRODUCTION

Dynamical systems are governed by initial value problems (IVP), which are,usually, nonlinear and frequently do not have a known solution (see Abarbanel and Sushchik (1993)). This article deal with two different approximation methods to nonlinear IVP solution, an analytical and a numerical method. Numerical methods are very efficient and provide approximations with the precision required, but it does not provide the needed insight to understand how a solution depends on the problem parameters. Alternatively, analytical approximations can often be found, particularly, if nonlinearity is relatively small, Wagg and Virgin (2012).

Perturbation methods are a powerful technique to compute analytical approximations to IVP, see Rao (1995), Simmonds and Mann (1986), Wagg and Virgin, 2012. In this paper we chosen to analyze a type of perturbation method called as Poincaré-Lindstedt method, which gives an uniformly valid asymptotic expansion for periodic solutions of weakly nonlinear oscillations, He (2002). This method assumes that the nonlinear IVP solution is a power series of the perturbation parameter, which will be introduced in the nonlinear equation. To compute the analytical approximation, this series should be truncated according to the number of terms desired. As a reference to estimate the quality of an approximation, a Runge-Kutta method is used for a specific value of the parameters. An cumulative error was determined to quantify the accuracy of the approximations. Afterwards, we analyzed how the number of terms influences the analytical approximations domain of validity. This contribution presents the Poincaré-Lindstedt methodology by applying it to the non-damped Duffing equation, the simplest nonlinear problem used in Mechanics, more details in Kovacic and Brennan (2011).

2 POINCARÉ-LINDSTEDT METHOD

In this section, the Poincaré-Lindstedt method, is briefly presented. Consider the general nonlinear equation

$$\ddot{u}(t) + \omega_0^2 u(t) + \alpha f(\dot{u}(t), u(t)) = 0, \tag{1}$$

with initial conditions $u(0) = A_0$ and $\dot{u}(0) = 0$, where $\dot{\square}$ means the derivation in relation to t, α is a constant parameter that controls the nonlinearity and f is function of u and $\dot{u}$. The first step of this technique is to introduce a perturbation parameter ϵ into the nonlinear term of the Eq. (1),

$$\ddot{u}(t) + \omega_0^2 u(t) + \epsilon\alpha f(\dot{u}(t), u(t) = 0\,. \tag{2}$$

Observe that when ϵ is equal to zero the equation becomes linear and, since its coefficients are constant, it has a well known analytical solution, see in Simmonds and Mann (1986). The second step is to change the time scale. A nondimensional parameter τ is defined as

$$\tau = \omega t\,, \tag{3}$$

where the angular frequency, ω, is function of the initial amplitude, A_0, given by

$$\omega(A_0) = \omega_0 + \epsilon\omega_1(A_0) + \epsilon^2\omega_2(A_0) + \dots\,, \tag{4}$$

where each ω_i, $i = 1, 2, 3, .., n$ is a function of A_0. Applying Eq. (3) into (2), we obtain

$$\omega^2 u''(\tau) + \omega_0^2 u(\tau) + \epsilon\alpha f(u'(\tau), u(\tau)) = 0\,, \tag{5}$$

where $\square'$ represents the derivation in relation to τ. This equation has as solution,

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots , \tag{6}$$

where each u_i, $i = 0, 1, 2, 3, .., n$ is an unknown function to be determined from the governing equation and the initial conditions. Applying the Eqs. (4) and (6) into (5), the terms with coefficients of same powers must be collected in order to form a family of linear initial value problems. These linear initial value problems must be solved according to the increasing order of powers ϵ. It is important to remark that as we are interested in periodic approximations, for each initial value problem the secular terms must be eliminated (the coefficients of secular term should match to zero). Doing this, it is possible to compute approximation to each ω_i. The last step is to replace each u_i and ω_i calculated into Eqs. (4) and (6). Some examples can be found in Pasquetti (2008), He (2002) and Wagg and Virgin (2012).

3 ANALYTICAL APPROXIMATION TO THE SOLUTION OF THE DUFFING EQUATION

To illustrate the application of the Poincaré-Lindstedt method described previously, consider $f(\dot{u}, u) = u^3$, Eq. (1) becomes

$$\ddot{u} + \omega_0^2 u + \alpha u^3 = 0 , \tag{7}$$

this equation is known as non-damped Duffing equation. Applying the first two steps mentioned previously, i. e, introducing a parameter ϵ multiplying the nonlinear term and, changing the time scale according to Eq. (3), Eq. (7) becomes

$$\omega^2 u'' + \omega_0^2 u + \epsilon\alpha u^3 = 0 . \tag{8}$$

To exemplify the Poincaré-Lindstedt method, we will show how to calculate the approximation of the solution to Eq. (8) considering three terms in the power series of ϵ given in Eqs. (4) and (6). These power series truncated become

$$u \approx u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 , \tag{9}$$

$$\omega \approx \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 . \tag{10}$$

Substituting Eqs. (9) and (10) into (8) and grouping the terms according to the power of ϵ, we have the following family of IVP

$$\omega_0^2 u_0'' + \omega_0^2 u_0 = 0 , \tag{11}$$

$$\omega_0^2 u_1'' + \omega_0^2 u_1 = -\alpha u_0^3 - 2\omega_0\omega_1 u_0'' , \tag{12}$$

$$\omega_0^2 u_2'' + \omega_0^2 u_2 = -\omega_1^2 u_0 - 2\omega_0\omega_1 u_1'' - 2\omega_0\omega_2 u_0'' - 3\alpha u_1 u_0^2 , \tag{13}$$

$$\omega_0^2 u_3'' + \omega_0^2 u_3 = -\omega_1^2 x_1'' - 2\omega_0\omega_1 u_2'' - 2\omega_0\omega_2 x_1'' - 2\omega_0\omega_3 u_0'' - 2\omega_1\omega_2 x_0'' - 3\alpha u_0 u_1^2 - 3\alpha u_0^2 u_2 , \tag{14}$$

with initial conditions $u_0(0) = A_0$, $u_0'(0) = 0$ and $u_i(0) = u_i'(0) = 0$, $i = 1, 2, 3$.

The solution of Eq. (11) according to the initial condition for u_0 and u_0' is $u_0(\tau) = A_0 \cos\tau$. Substituting it into equation (12), we obtain

$$\omega_0^2 u_1'' + \omega_0^2 u_1 = \left(2A_0\omega_0\omega_1 - \frac{3A_0^3\alpha}{4}\right)\cos\tau - \frac{A_0^3\alpha}{4}\cos 3\tau \tag{15}$$

observe that the right side of this equation contains a term $\cos\tau$, which causes resonance. To avoid this, the coefficients from this term should be matching to zero in a way that the term is eliminated. Doing this, we find

$$\omega_1 = \frac{3A_0^2\alpha}{8\omega_0}\,. \tag{16}$$

After the elimination of the resonant terms, the Eq. (15) can be solved to u_1 and subsequently replaced in Eq. (13), which becomes

$$\begin{aligned}\omega_0 u_2'' + \omega_0 u_2 = &\left(\frac{\cos(3\tau)}{4} + \frac{3\cos(\tau)}{4}\right)\left(\frac{3A_0^5\alpha^2}{8\omega_0^2} + \frac{9A_0^3\alpha\omega_1}{4\omega_0}\right)\\ &+ \cos(\tau)\left(A_0\omega_1^2 + 2A_0\omega_0\omega_2 - \frac{7A_0^3\alpha\omega_1}{4\omega_0}\right)\\ &- \left(\frac{3A_0^5\alpha^2}{8\omega_0^2}\right)\left(\frac{5\cos(3\tau)}{16} + \frac{\cos(5\tau)}{16} + \frac{5\cos(\tau)}{8}\right),\end{aligned} \tag{17}$$

the same steps used to solve Eq. (15) should be done to solve Eq. (17). Substituting all the terms u_i and ω_i calculated into Eqs. (9) and (10), we obtain an analytical approximation to u as function of τ. If desired, the approximation can be written as function of t using the relation given in Eq. (3).

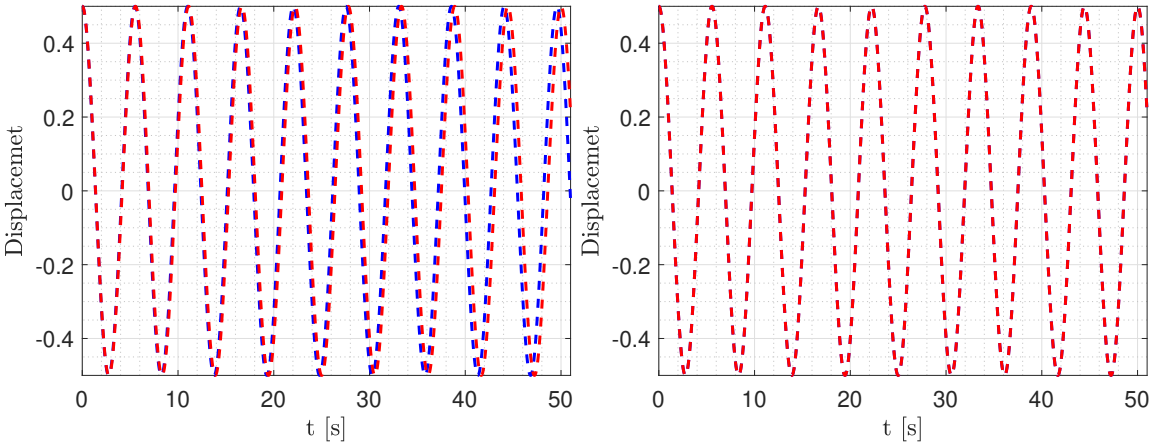
Observing the Eq. (17), it is possible to verify that this equation has more terms that Eq. (15), consequently, to solve it is more irksome. This happen because we are solving an equation that corresponds to a higher order of ϵ. Table 1 shows the relationship between order of series ϵ, the number of equations in the family of IVP, number of terms in the right side of each equation and the total of terms in right side of equations. Looking at Tab. 1, we observe that the greater the order of series ϵ, the greater the quantity of equations to solve and total of terms. Since approximations with high order are cumbersome to solve by hand, a routine in MATLAB software using symbolic algebra was developed. This routine allows the computation of approximations to solution of u according to the numbers of terms chosen by the user.

Order of series ϵ	Number of equations in the family of IVP	Number of terms in the right side of each IVP	Total of terms in right side
1	2	0	2
		2	
2	3	0	6
		2	
		4	
3	4	0	13
		2	
		4	
		7	
4	5	0	24
		2	
		4	
		7	
		11	
5	6	0	39
		2	
		4	
		7	
		11	
		15	

Table 1: Relationship between order of series ϵ, the number of equations in the family of IVP, number of terms in the right side of each equation and the total of terms in right side of equations.

4 DISCUSSION OF THE RESULTS

The routine implemented in MATLAB software allows the computation of approximations to the displacement, velocity and frequency, ω, with desired order and initial conditions. Another routine, also implemented in MATLAB, was developed to calculate numerical approximations to u through Runge-Kutta method, using the command ODE45. The obtained analytical approximations are evaluated for different combinations of parameters values and compared with the numerical approximation to understand their domain of validity. A primary result around the importance of the approximation order can be observed Fig. 1. In Fig. 1(a) it is shown an approximation of u considering just one term and, in Fig. 1(b) considering six terms. Observe that with one term, for t smaller than 20, the analytical approximation has a good accuracy compared with the numerical, however, as t increases, the curves begin to separate. With six terms, we observe that the two curves are almost indistinguishable for $t < 50$.



(a) (b)

Figure 1: Numerical (red) and analytical approximation (blue) (a) with one term; (b) with six term; with $\epsilon = 1$, $\alpha = 1.5$, $\omega_0 = 1$ e $A_0 = 0.5$.

To improve the study about the domain of validity of the approximations, we use two different measures of error. The first error used is the module of the maximum difference between the analytical and numerical approximations during a time interval $[0, T]$ for fixed parameters values, that is

$$\max_{0<t<T} |u^a(t) - u^n(t)|. \tag{18}$$

Since system trajectory is periodic, their maximum amplitude has the same value of the initial displacement, A_0. Using this metric, there is a maximum error which happens when, simultaneously, one approximation are in the maximum displacement and the other are in the minimum, as exemplified by Fig. 2, therefore the maximum error value is $2A_0$.

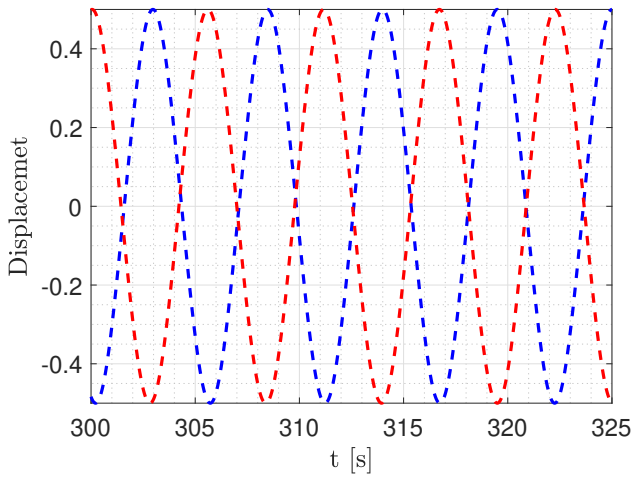
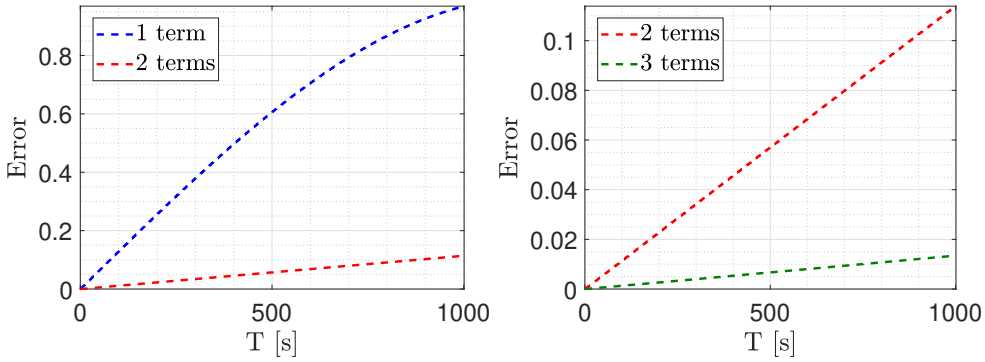


Figure 2: Example: error maximum in the displacement

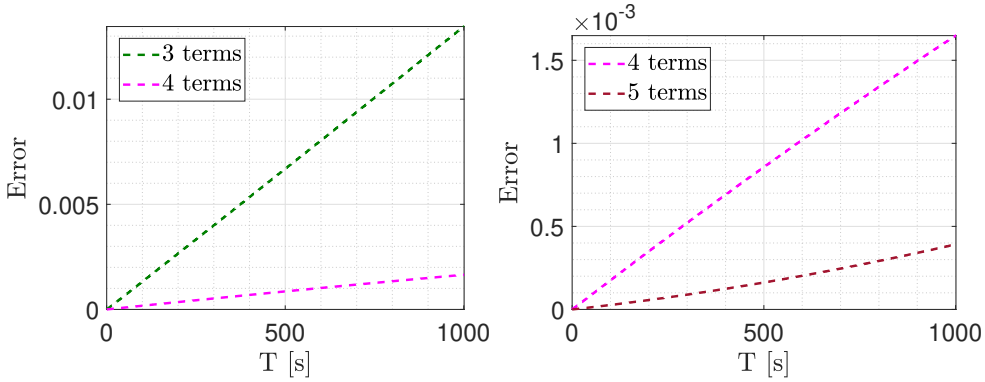
Figure 3a shows the approximation errors with one and two terms as function of T. Observe that with one term, the maximum error is achieved when $T \approx 1000$, while with two terms, the

error around 0.1 when $T = 1000$. Figure 3 shows, as aforementioned, that the increase the order of the approximation improves the domain of validity.



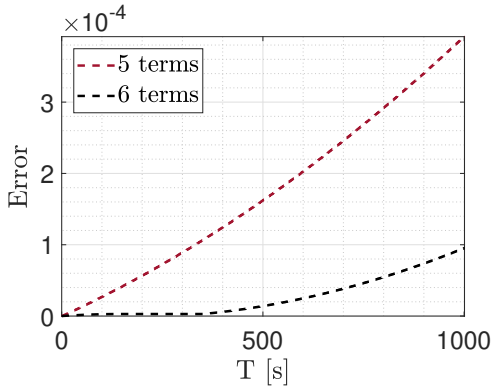
(a) Noncumulative error between 1 and 2 terms.

(b) Noncumulative error between 2 and 3 terms.



(c) Noncumulative error between 3 and 4 terms.

(d) Noncumulative error between 4 and 5 terms.



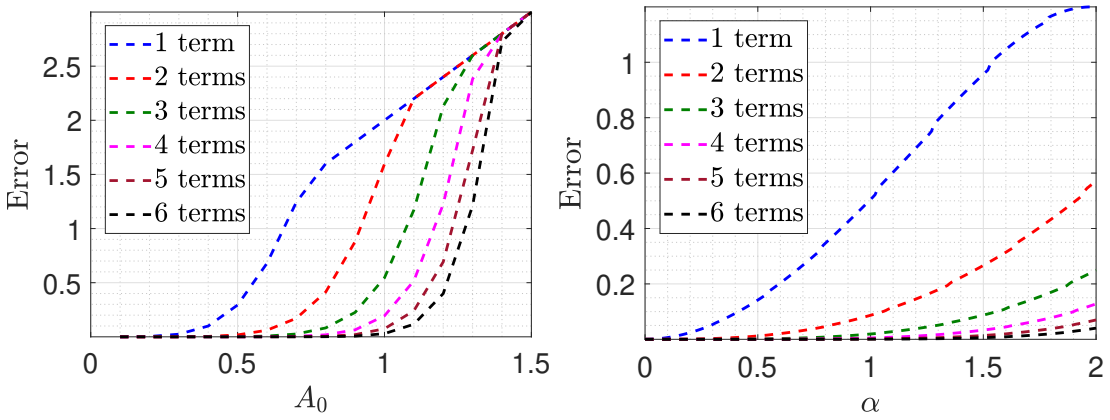
(e) Noncumulative error between 5 and 6 terms.

Figure 3: Noncumulative error between the analytical and numerical approaches, considering $\alpha = 0.75$, $\omega_0 = 1$, $A_0 = 0.5$ and $\epsilon = 1$.

Since the dynamics of nonlinear systems are influenced by the initial conditions, we analyzed the influence of the initial displacement, A_0, in the domain of validity using the metric given by Eq.(18). The outcome is shown in Fig. 4a. As expected, the results show us that when the initial

displacement grows, the order of approximation also should grows to keep the same error. It is important to observe that the error of the first approximation is always equal or bigger than the others errors obtained with higher order approximations. For example, when $A_0 \approx 0.5$ the error with one term is about 0.3, while the errors of the others approximation are less than 0.02. When $A_0 = 1.0$, while the error of approximation with one term already reaches the maximum error, i. e., twice the initial displacement value, the error of approximation with two terms is lower, however is somehow close to it if compared with the errors of the approximation with four, five and six order, which presents errors smaller than 0.2.

Another analysis was to observe the behavior of the error according to the variation of α, the parameter that controls the nonlinearity, similar to what was made in relation to A_0. The results are shown in Fig.4b. Observe that with the increase of α, the curves that represent the lower order approximation reach the maximum value before than the others (recalling that the maximum error is twice the initial displacement, $2A_0$). These results show again that the first order error is always equal or bigger than the others errors with higher order. In the performed analysis, the maximum value of αconsidered is 2.0 since this can be considered a high value to a parameter that controls the nonlinearity.



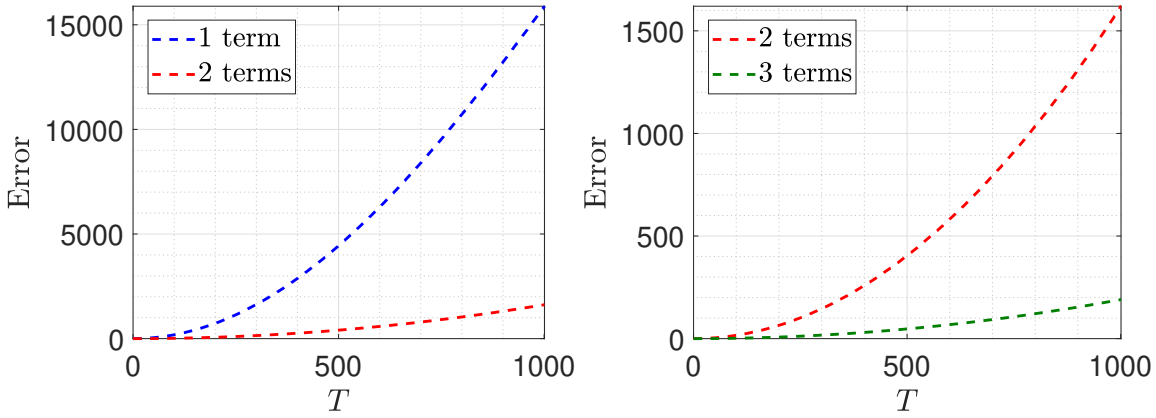
(a) Variation of A_0. (b) Variation of α.

Figure 4: (a) Variation of $0.1 < A_0 < 1.5$ with $\epsilon = 1$, $\alpha = 0.5$ and $\omega_0 = 1$. (b) Variation of $0 < \alpha < 2$ with $\epsilon = 1$, $A_0 = 0.6$ and $\omega_0 = 1$.

To avoid the maximum error that can happen when we use the error given by in Eq. (18), we considered in the analysis another measure of error, which we can be seen as measure of the cumulative error. This cumulative error is given by the sum of the module of the maximum difference between the analytical and numerical approximation during a time interval $[0, T]$ for fixed parameters values, that is

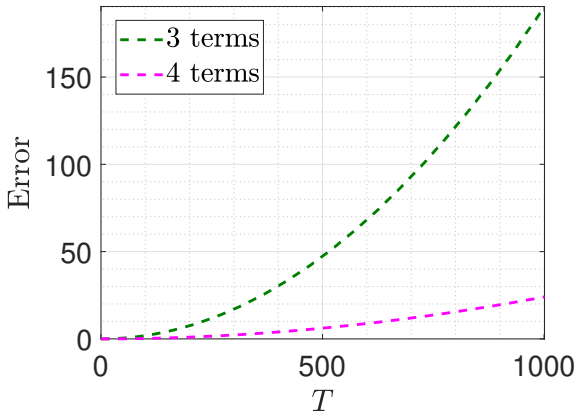
$$\sum_{0<t<T} \max_{0<t<T} \left| u^a(t) - u^n(t) \right|, \tag{19}$$

where u^a and u^n are the analytical and numerical approximations respectively. Figure 5 shows this error for the approximations until sixth order. Observing it, it is possible to verify that at the end of the interval, while the error obtained with the first order approximation is around 15000, the error obtained with the second order is around 1500 and, with the sixth order is lower than 1. This shows that the increase of the order approximation, improves the domain of validity.

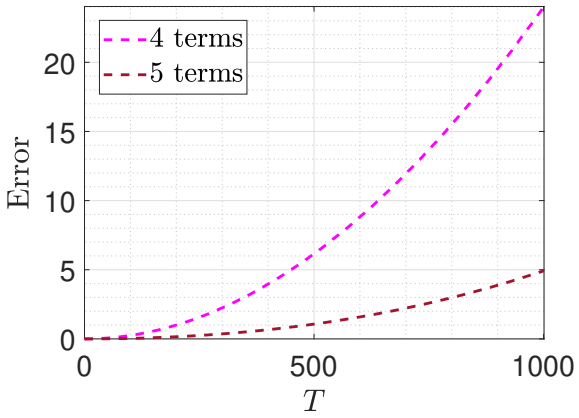


(a) Cumulative error between 1 and 2 terms.

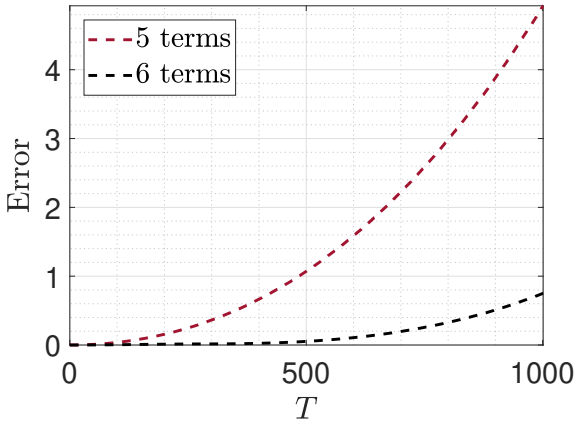
(b) Cumulative error between 2 and 3 terms.



(c) Cumulative error between 3 and 4 terms.



(d) Cumulative error between 4 and 5 terms.



(e) Cumulative error between 5 and 6 terms.

Figure 5: Cumulative error between the analytical and numerical approaches, considering $\alpha = 0.75$, $\omega_0 = 1$, $A_0 = 0.5$ and $\epsilon = 1$.

5 CONCLUSIONS

Approximations with high order are cumbersome to solve by hand, and to support the analytical computation, a MATLAB routine was implemented using symbolic algebra. This routine allows to compute approximations to displacement, velocity and frequency according to the numbers of terms chosen by the user. In this paper, we calculated analytical approximations un-

til sixth order. These expressions were used in another routine also implemented in MATLAB, which assigns values to the parameters of analytical approximations and then, compare them with numerical approximations. To quantify the difference between these approximations, two metrics of error were used, a noncumulative and a cumulative one. First, we used the noncumulative. However, as we verified that this metric presents a maximum error, $2A_0$, exemplified in Fig. 2, we used the second one, i.e., the cumulative. In both analysis, the results obtained show that the domain of validity is influenced by the number of terms considered in the analytical approximation.

ACKNOWLEDGEMENTS

The authors acknowledge the support given by FAPERJ, CNPq and CAPES.

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