Mecánica Computacional Vol. XXIV
A. Larreteguy (Editor)
Buenos Aires, Argentina, Noviembre 2005

FREE VIBRATION OF NONSYMMETRICAL THIN WALLED BEAMS. NUMERICAL-EXPERIMENTAL COMPARISON

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Key words: Coupled Flexural-torsional Vibrations, Vlasov Theory, Natural Frequencies, Dynamic Tests.

Abstract. *Thin walled and open section beams are extensively used as structural components in different structures in Civil, Aeronautical and Mechanical Engineering fields. Free vibrations of doubly symmetrical beams or beams with one axis of symmetry are widely studied, in general by using Bernoulli-Navier theory. However, results about doubly unsymmetrical beams are rather limited. In this case, triple coupled flexural-torsional vibrations are observed.*
In this paper, a numerical-experimental comparison is presented about natural frequencies of doubly unsymmetrical thin-walled and open cross-section beams. The equations of motion are based on Vlasov's theory of thin-walled beams, which is modified to include the effects of shear flexibility, rotatory inertia in the stress resultants and variable cross-sectional properties. The formulation is also applicable to solid beams, constituting therefore a general theory of coupled flexure and torsion of straight beams. The differential equations are shown to be particularly suitable for analysis in the frequency domain using a state variables approach. A discussion related to vibration of unsymmetrical channel cross-section beams is presented. In this sense, relevant topics discussed in recent works are pointed out and clarified. Experimental tests about these unsymmetrical beams are presented which allow verify the theory presented in this paper and provide good quality data that can be used for checking the accuracy and reliability of different theories

1 INTRODUCTION

Thin walled and open section beams are extensively used as structural components in different structures in Civil, Aeronautical and Mechanical Engineering fields. Free vibrations of doubly symmetrical beams or beams with one axis of symmetry are widely studied, in general by using Bernoulli-Navier theory. However, results about doubly unsymmetrical beams are rather scarce in the literature, especially those related with experimental evidences. In this case, triple coupled flexural-torsional vibrations are observed.

The determination of natural frequencies and modes of vibration of undamped continuous beams and shafts is discussed in detail in Pestel and Leckie[1] who also describe the calculation of dynamic response to harmonic excitation. Ebner and Billington[2] employed numerical integration to study steady state vibrations of damped Timoshenko beams. Numerous other applications can be found in the literature concerning straight and curved beams, as well as arch and shell structures. On the other hand, the theory formulated by Vlasov[3] has been extensively used in the dynamic analysis of thin-walled, open section beams, as exemplified by the studies of Christiano and Culver[4] and Yoo and Fehrenbach[5], in which theoretical predictions of natural frequencies and displacements induced in curved and straight beams by travelling loads, closely match experimentally determined values. In slender beams, modelling the structure with a large number of finite shell elements or other more refined approaches does not lead to any improvement in the correlation with experimental results, because the differences between the theoretical models are usually smaller than the experimental errors.

Nevertheless, although Vlasov's theory for open section beams is already firmly established, it presents some limitations, namely: a) as in the common Bernoulli theory for flexure, it is assumed that shear strains do not contribute to the beam flexibility. Consequently, important errors should be expected in the analysis of deep beams or in the dynamic response associated to higher vibrations modes, even in the case of slender beams (Timoshenko and Young[6]). b) The influence of rotatory inertia in the stress resultants is also neglected and c) Vlasov's fourth-order equations are valid only for beams with uniform cross-section. In previous papers[7, 8], the author proposed a modified theory, which is based on Vlasov's formulation, but it accounts the effects mentioned above. This formulation, using the so-called state variables approach in the frequency domain, lends itself to efficient numerical treatment, which on account of generality and precision can be very useful in a variety of applications.

Other theories that also account for coupling between bending and torsion in beams are presented in Gere and Lin[9] that derive a simplified equation for uniform open section beams and Muller[10] that formulates a general theory that include all coupling effects between the equations of motions, but it is not easy to handle in applications. Most other contributions in the field are restricted to particular cases. For example, Aggarwal and Cranch[11] and Yaman[12] deal with channel-section beams and Ali Hasan and Barr[13] with equal angle-sections.

More recently Tanaka and Bercin[14] extend the approach of Bishop et al.[15] to study triply coupling of uniform beams using Mathematica. The governing differential equations presented by the mentioned authors[14] exhibit a confusion of co-ordinate system that was clarified by Arpaci and Bozdag[16]. However, the equations presented in the last paper[16] correcting those presented in[14] are verified comparing results with a case that neglects the same terms of Tanaka and Bercin[14] making it impossible to verify the accuracy of the both theories. In this paper, a numerical study based on the equations developed in previous papers[7,8] is presented and the results of Arpaci and Bozdag[16] are discussed. Moreover, the field matrix of the state variables approach is presented for the case in which the warping is neglected. These equations lead to a more simple theory that is used for comparison purposes in this paper.

The above discussion demonstrates that it is essential to compare with experimental results in order to obtain reliable conclusions about the accuracy and applicability of different theories. For these reasons, a set of experimental test were conducted and the results are presented in this paper because it is the way in which the results obtained by different theories should be verified. These experimental results can be used for checking the accuracy and reliability of calculation methods and procedures.

2 THEORY

2.1 Equations of motion

Following Vlasov's convention, the left-handed rectangular global coordinates system (*x*, *y*, *z*) shown in Figure 1 was adopted. The associated displacements are designated ξ, η, and ζ. The basic concepts needed to introduce the effects of shear strains, rotatory inertia and variable cross-sectional properties within the framework of Vlasov's theory are described by the author in previous papers[7,8], in which a complete derivation of the equations of motion for free and forced vibrations may be found, as well as comparisons with other continuum formulations and a thorough discussion of the definition of shear coefficients. The present paper is more oriented to practical applications and only the derivation of the differential equations for free vibrations in the state variables method is given.

In Figure 1, *A* represents the centroid and *O* the shear center. For the case of free vibrations, the physical model is formed by the following three fourth order partial differential equations in the generalised displacements ξ, η, and θ:

$$E\left[J_y(z)\left(\frac{\partial^4\xi}{\partial z^4}-\frac{\partial^3\gamma_{mx}}{\partial z^3}\right)+2\left(\frac{\partial^3\xi}{\partial z^3}-\frac{\partial^2\gamma_{mx}}{\partial z^2}\right)\frac{dJ_y(z)}{dz}\right]-\rho J_y(z)\left(\frac{\partial^4\xi}{\partial z^2\partial t^2}-\frac{\partial^3\gamma_{mx}}{\partial z\partial t^2}\right)-$$
$$-\rho\frac{dJ_y(z)}{dz}\left(\frac{\partial^3\xi}{\partial z\partial t^2}-\frac{\partial^2\gamma_{mx}}{\partial t^2}\right)+\rho F_T(z)\left(\frac{\partial^2\xi}{\partial t^2}+a_y\frac{\partial^2\theta}{\partial t^2}\right)=0 \tag{1a}$$

$$E\left[J_x(z)\left(\frac{\partial^4\eta}{\partial z^4}-\frac{\partial^3\gamma_{my}}{\partial z^3}\right)+2\left(\frac{\partial^3\eta}{\partial z^3}-\frac{\partial^2\gamma_{my}}{\partial z^2}\right)\frac{dJ_x(z)}{dz}\right]-\rho J_x(z)\left(\frac{\partial^4\eta}{\partial z^2\partial t^2}-\frac{\partial^3\gamma_{my}}{\partial z\partial t^2}\right)-$$
$$-\rho\frac{dJ_x(z)}{dz}\left(\frac{\partial^3\eta}{\partial z\partial t^2}-\frac{\partial^2\gamma_{my}}{\partial t^2}\right)+\rho F_T(z)\left(\frac{\partial^2\eta}{\partial t^2}+a_x\frac{\partial^2\theta}{\partial t^2}\right)=0 \tag{1b}$$

$$E\left[J_{\varphi}(z)\frac{\partial^4\theta}{\partial z^4}+2\frac{\partial^3\theta}{\partial z^3}\frac{dJ_{\varphi}(z)}{dz}\right]-\rho J_{\varphi}(z)\frac{\partial^4\theta}{\partial z^2\partial t^2}-\rho\frac{dJ_{\varphi}(z)}{dz}\frac{\partial^3\theta}{\partial z\partial t^2}+$$
$$+\rho F_T(z)\left(a_y\frac{\partial^2\xi}{\partial t^2}-a_x\frac{\partial^2\eta}{\partial t^2}+r^2\frac{\partial^2\theta}{\partial t^2}\right)-GJ_d(z)\frac{\partial^2\theta}{\partial z^2}-G\frac{dJ_d(z)}{dz}\frac{\partial\theta}{\partial z}=0 \tag{1c}$$

In these equations, F_T is the cross-sectional area, J_x and J_y are the second moments of area of the cross-section in relation to the centroidal principal axes, J_φ the sectorial second moment of area, J_d the torsion modulus, a_x and a_y the coordinates of the shear centre. ρ denotes the mass density of the beam material. E and G are the Young's and the shear modulus respectively. Finally, γ_{mx} and γ_{my} represent the mean values of shear strains over a cross-section z = constant and

$$r^2=a_x^2+a_y^2+\frac{J_x+J_y}{F_T} \tag{2}$$

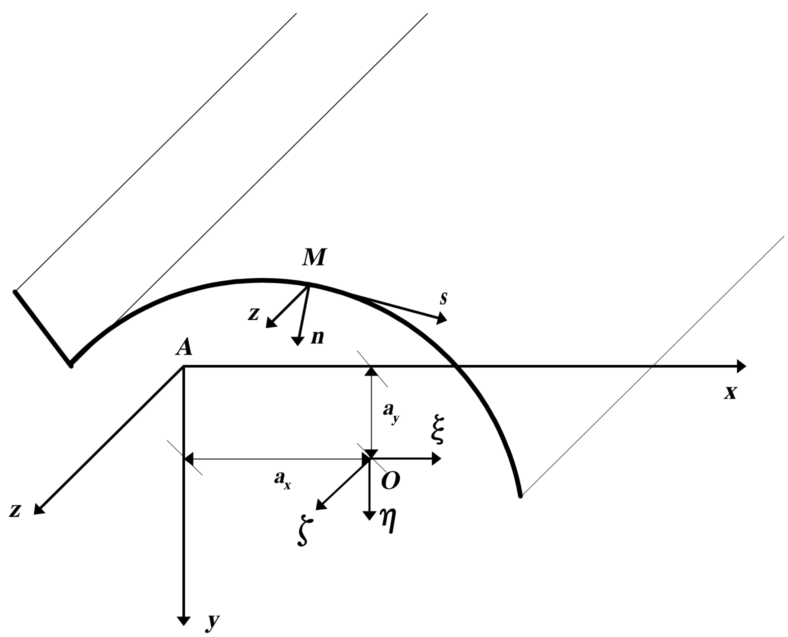


Figure 1: Definition of terms

The system (1) represents a general model of non-uniform beams that take into account triply coupled flexural-torsional vibrations. It must be pointed out that the longitudinal vibration equation related to the generalised displacement ζ (Figure 1) is non-coupled with the rest of the system (1) and it was not taken in consideration in the analysis. In the case that the longitudinal vibrations are of interest, this equation can be treated independently

2.2 State variables approach

Using the Fourier transform, an equivalent system with twelve first order partial differential equations with twelve unknowns, in the frequency domain, is obtained. The scheme described above is known in the literature as 'state variables approach'. Six geometric and six static unknown quantities are selected as components of the state vector v: The displacements ξ and η, the bending rotations ϕ_x and ϕ_y, the normal shear stress resultants Q_x and Q_y, the bending moments M_x and M_y, the torsional rotation θ and its spatial derivative θ', the total torsional moment M_T and the bimoment B.

$$\mathbf{v}(z,\omega) = \{\eta, \phi_y, Q_y, M_x, \xi, \phi_x, Q_x, M_y, \theta, \theta', M_T, B\}^{\mathrm{T}} \tag{3}$$

$$M_T = H_\varphi + H_k \tag{4}$$

with $H_k = GJ_d\theta'$ = Saint Venant torsion moment. The system is:

$$\frac{\partial \mathbf{v}}{\partial z} = \mathbf{A}\mathbf{v} \tag{5}$$

In which **A** is the system matrix given by:

$$\mathbf{A} = \left[\begin{array}{cccc|cccc|cccc}
0 & 1 & \frac{1}{k'_y FG} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{EJ_x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\rho F\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho F\omega^2 a_x & 0 & 0 & 0 \\
0 & \rho J_x\omega^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{k'_x FG} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{EJ_y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\rho F\omega^2 & 0 & 0 & 0 & -\rho F\omega^2 a_y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\rho J_y\omega^2 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{EJ_\varphi} \\
\rho F\omega^2 a_x & 0 & 0 & 0 & -\rho F\omega^2 a_y & 0 & 0 & 0 & -\rho F\omega^2 r^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B\theta' & 1 & 0
\end{array}\right] \tag{6}$$

in which k'_x and k'_y denote the Cowper's shear coefficients and:

$$B\theta' = \rho J_{\varphi}\omega^2 - GJ_d \tag{7}$$

The components of the 12-dimensional state vector **v** are designated "state variables". In the frequency domain, the state variables depend on the frequency ω and the longitudinal coordinate *z*. For simplicity, the same notation is being used for the state variables and their Fourier transforms, since the domain can usually be identified by the indication of the function arguments. For example, $\eta(z, t)$ and $\eta(z, \omega)$ refer to the *y*-displacement in the time domain and to its Fourier transform, respectively. It is important to note that the present formulation constitutes a general theory of beams applicable to solid as well as thin-walled beams.

2.3 Numerical procedure and boundary conditions

The system (5) may be easily integrated using standard numerical procedures, such as the fourth order Runge-Kutta method, the predictor-corrector algorithm or other approaches. In order to solve the two-point value problem encountered both in the determination of natural frequencies and in dynamic response calculations, the latter must be transformed to an initial value problem as shown, for example, by Ebner and Billington[2]. The procedure is normally applied in the transfer matrix method (Pestel and Leckie[1]). Natural frequencies are determined by means of the well-known Thomson's method.

The classical boundary conditions are considered in this paper: clamped, free or simply supported.

Clamped Boundary

$$\xi = \eta = 0,\ \phi_x = \phi_y = \theta = 0,\ \theta' = 0 \tag{8}$$

Free Boundary

$$Q_y = Q_x = 0,\ M_x = M_y = M_T = 0,\ B = 0 \tag{9}$$

Hinged Boundary

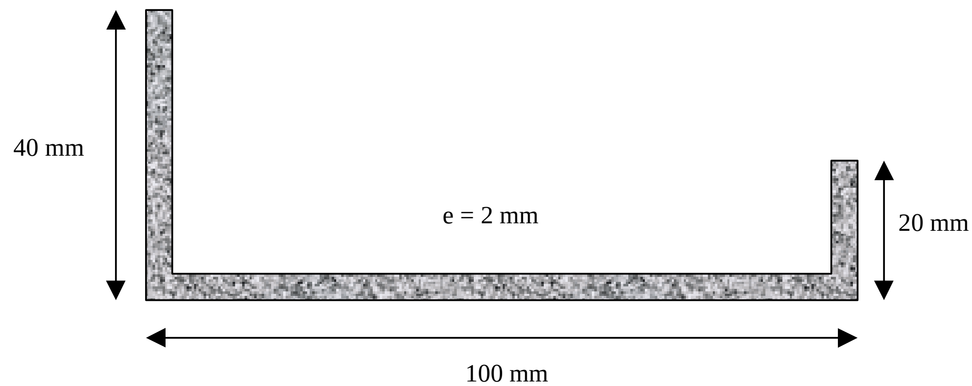
$$\xi = \eta = 0,\ M_x = M_y = 0,\ \theta = 0,\ B = 0 \tag{10}$$

3 DYNAMIC TESTS

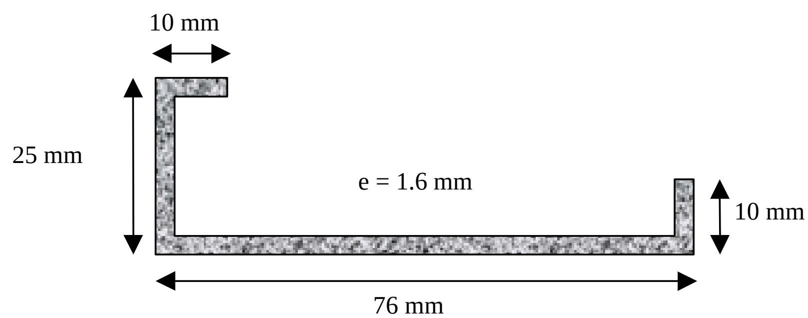
3.1 Case study

Sometimes to compare numerical results obtained with different theories leads to an undefined problem and another point of view is necessary in order to solve differences and show the accuracy of these theories. As it was pointed out in the point 1, experimental evidences about doubly unsymmetrical beams are rather scarce in the literature. Moreover, taking into account the controversial point discussed in this paper, series of free vibration test were carried out about doubly unsymmetrical aluminum beams.

Aluminum beams with cross sections shown in Figure 2 were tested. Two different lengths were applied: 2m (Beams 1 and 2) and 1.5m (Beams 3 and 4). Moreover, two support conditions were used: Free-fixed and fixed-fixed.



a) Beams 1 and 3



b) Beams 2 and 4

Figure 2. Cross-section of the beams tested

3.2 Experimental set-up and instrumentation

Three accelerometers KYOWA AS-GB were used to measure the dynamic response of the beams. A dynamic strain amplifier KYOWA DPM-612B amplified the signal generated by the accelerometers. Moreover, the amplifier have a low-pass filter applied to avoid aliasing. A data acquisition board Computerboards PCM-DAS16D/16 of 16 bit of resolution and a maximum conversion time of 10 μs (100 KHz) was mounted on a notebook computer in order to record and process the signals by means of the program HP VEE 5.0[17]. A global view of one test is presented in Figure 3. The beams were excited with a hammer blow in different points in order to excite different modes (see Figure 4).

The signals were sampled, for both beams, with the following parameters: N = 2500 (total number of points for each channel), n = 500 (sampling rate or number of points per second), T = 5 s (total time of the sample), Δt = 0.002 s, (time interval), Δf = 0.2 Hz (frequency interval), $f_{máx}$ = 250 Hz (maximum frequency).

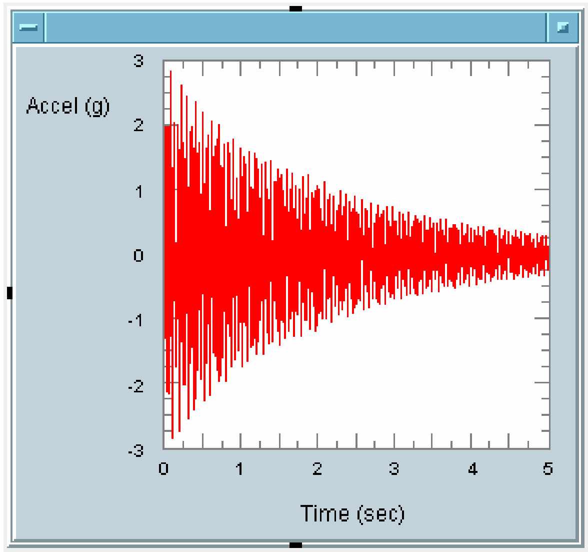
An algorithm to obtain and process the data was programmed in the environment HPVEE[17]. After applying the Fast Fourier Transform, the spectrum was calculated using the Welch method (Peeters[18]).

Figure 3: Experimental set-up and instrumentation

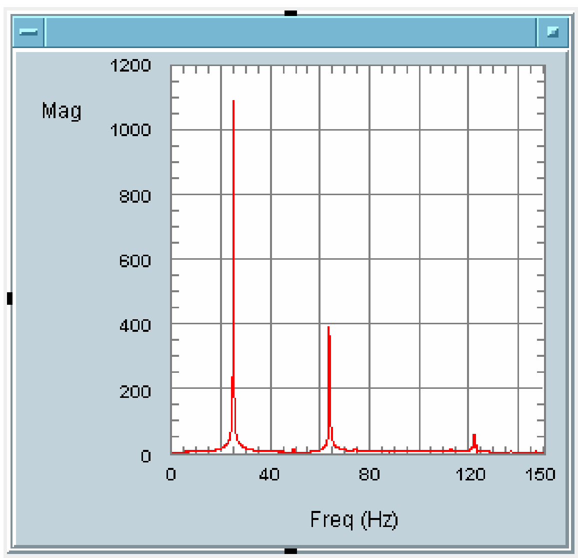
Figure 4: Impulse loading

3.3 Dynamic response and experimental results

In order to carry out the system identification and determination of natural frequencies, the peak-picking method was used[19]. This method has enough accuracy for this type of elements. Procedures and filters to avoid aliasing and leakage were applied[19]. As an example, accelerations registered by a transducer and the corresponding frequency spectrum are presented in Figure 5 for the case of Beam 1 fixed-fixed.



a) Waveform (Time)



b) Magnitude spectrum

Figure 5: Dynamic response measured

Then, the geometric and measured mechanical properties used in order to apply the theory presented in this paper are:

$l = 2.00$ m $F_T = 3.2\ 10^{-4}$ m^2
$J_x = 3.024\ 10^{-8}$ m^4 $J_y = 4.595\ 10^{-7}$ m^4
$J_{xy} = 0$
$J_\varphi = 2.838\ 10^{-11}$ m^6 $J_d = 4.270\ 10^{-10}$ m^4
$a_x = 2.67\ 10^{-2}$ m $a_y = -1.30\ 10^{-2}$ m
$E = 4.50\ 10^{10}$ N/m^2 $\rho = 2650$ kg/m^3
$\nu = 0.25$

The experimental results are showed in Table 1. In order of take into account the random errors characterized were calculated the estimators of the mean value μ, the standard deviation σ and the coefficient of variation *CV*.

Table 1: Experimental Results. Frequency (Hz).

	Clamped-Free					Clamped-Clamped				
	f_1	f_2	f_3	f_4	f_5	f_1	f_2	f_3	f_4	f_5
BEAM 1	5.2	12.6	25.8	28.4	69.4	24.5	49.1	63.5	122.5	196.4
BEAM 2	3.8	11.2	20.2	43.2	69.2	16.1	32.1	39.1	70.4	111.3
BEAM 3	8.8	17.8	45.8	108.2	-	39.0	78.0	104.4	116.8	142.8
BEAM 4	6.2	18.0	29.2	59.2	116.8	24.0	59.0	100.7	124.6	-

4 NUMERICAL RESULTS

4.1 Numerical model

A numerical tool was developed using the program with the equations and theory presented in point 2 (Program DYBEAM[7,8]). For the case studied, the definition of axes and shear centre coordinates is given in Figure 6.

Then, the geometric and mechanical properties used in order to apply the theory presented in this paper are:

a) Beams 1 and 3.

$l = 2.00$ m (Beam 1) $l = 1.50$ m (Beam 3)
$F_T = 3.2\ 10^{-4}$ m^2

$J_x = 3.024\ 10^{-8}$ m^4 $\qquad$ $J_y = 4.595\ 10^{-7}$ m^4
$J_\varphi = 2.838\ 10^{-11}$ m^6 $\qquad$ $J_d = 4.270\ 10^{-10}$ m^4
$a_x = 2.67\ 10^{-2}$ m $\qquad$ $a_y = -1.30\ 10^{-2}$ m
$E = 4.50\ 10^{10}$ N/m^2 $\qquad$ $\rho = 2650$ kg/m^3
$\nu = 0.25$

b) Beams 2 and 4.

$l = 2.00$ m (Beam 2) $\qquad$ $l = 1.50$ m (Beam 4)
$F_T = 1.891\ 10^{-4}$ m^2
$J_x = 9.19\ 10^{-9}$ m^4 $\qquad$ $J_y = 1.42\ 10^{-7}$ m^4
$J_\varphi = 2.522\ 10^{-12}$ m^6 $\qquad$ $J_d = 1.610\ 10^{-10}$ m^4
$a_x = -2.77\ 10^{-2}$ m $\qquad$ $a_y = -0.74\ 10^{-2}$ m
$E = 4.20\ 10^{10}$ N/m^2 $\qquad$ $\rho = 2650$ kg/m^3
$\nu = 0.25$

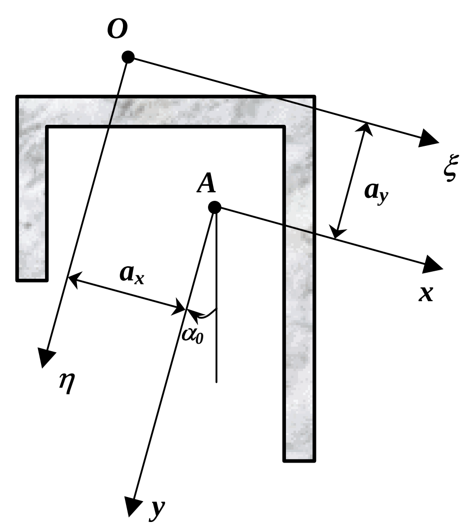


Figure 6: Principal axes of channel unnsymetrical beams.

4.2 Finite element model

In order to have an additional comparison of the theory proposed in this paper, for the case of doubly unsymmetrical thin walled beams, the results for the cantilever beam are compared with a finite element solution. The finite element software SAP2000N[20] was used in this comparison, modeling the beam by means of 20 rectangular Shell elements of four nodes. The same mechanical properties were used in this case. For comparison purposes, in Figure 7, a particular mode shape obtained is presented.

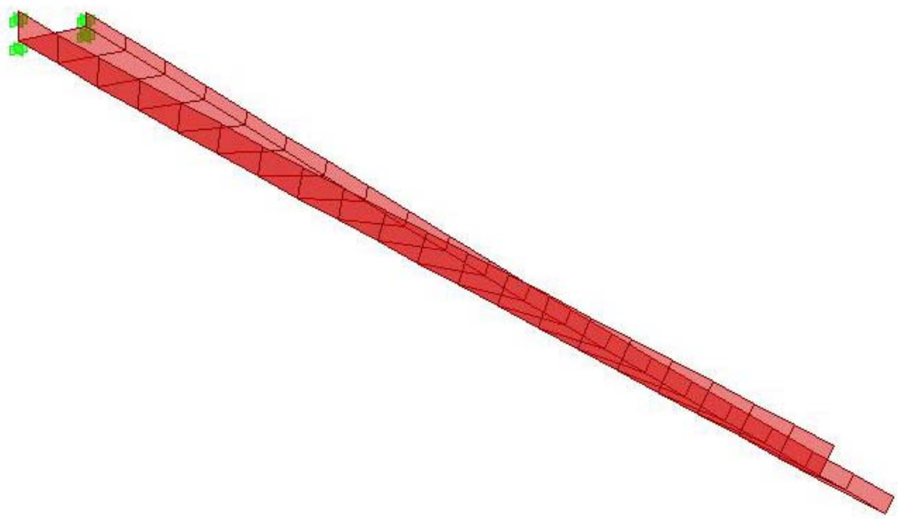


Figure 7: Mode 2. Beam 1. Clamped-Free. Software SAP2000.

4.3 Numerical-experimental comparison

In this section a comparative study of the results obtained in the experimental and numerical analysis will be presented. Due to the cross section of all beams are doubly asymmetrical, all vibration modes are flexural-torsional coupled modes. However, at this section, the modes are identified by the predominant motion.

a) Beam 1: Results are presented in Tables 2 and 3.

Table 2: Numerical-experimental comparison. Beam 1 clamped-free.

Frequency	Mode	Test Hz	DYBEAM[7,8] Freq. Hz	DYBEAM[7,8] Diff. %	SAP2000[20] Freq. Hz	SAP2000[20] Diff. %
1	1 Vertical flexural (Y)	5.2	5.3	1.9	5.15	-1.0
2	1 Torsional	12.6	12.0	4.8	9.9	21.4
3	1 Lateral flexural (X)	25.8	23.4	9.3	22.4	13.2
4	2 Vertical flexural (Y)	28.4	28.5	-0.4	26.5	6.7
5	2 Torsional	-	50.9	-	41.9	-
6	3 Vertical flexural (Y)	69.4	67.3	3.0	59.1	14.8

Table 3: Numerical-experimental comparison. Beam 1 clamped-clamped.

Frequency	Mode	Test Hz	DYBEAM[7,8] Freq. Hz	DYBEAM[7,8] Diff. %	SAP2000[20] Freq. Hz	SAP2000[20] Diff. %
1	1 Vertical flexural (Y)	24.5	24.8	-1.2	22.2	9.4
2	1 Lateral flexural(X)	49.1	47.9	2.4	40.5	17.5
3	2 Vertical flexural (Y)	63.5	63.4	0.2	55.2	13.1
4	3 Vertical flexural (Y)	122.5	119.1	2.8	102.8	16.1

It can be noted that DYBEAM gives a mean difference of 1.6% in flexural vertical modes, meanwhile SAP2000 gives a mean difference of 10.2% for the same modes. In connection with lateral flexural modes, DYBEAM gives a mean difference of 5.5% and SAP2000 gives a mean difference of 17.4%. Moreover, it can be observed the difficulties of experimental tests to capture the coupled lateral flexural-torsional modes.

b) Beam 2: Results are presented in Tables 4 and 5.

Table 4: Numerical-experimental comparison. Beam 2 clamped-free.

Frequency	Mode	Test Hz	DYBEAM[7,8] Freq. Hz	DYBEAM[7,8] Diff. %	SAP2000[20] Freq. Hz	SAP2000[20] Diff. %
1	1 Vertical flexural (Y)	3.8	3.7	2.6	3.8	0.0
2	1 Torsional	11.2	11.1	0.9	8.7	22.3
3	1 Lateral flexural (X)		16.1		16.6	
4	2 Vertical flexural (Y)	20.2	20.3	-0.5	19.6	3.0
5	3 Vertical flexural (Y)	69.2	70.4	1.7	64.5	4.7

Table 5: Numerical-experimental comparison. Beam 2 clamped-clamped.

Frequency	Mode	Test Hz	DYBEAM[7,8] Freq. Hz	DYBEAM[7,8] Diff. %	SAP2000[20] Freq. Hz	SAP2000[20] Diff. %
1	1 Vertical flexural (Y)	16.1	16.15	-0.3	15.2	5.6
2	1 Lateral flexural (X)	32.1	37.9	-18.1	33.4	-4.0
3	2 Vertical flexural (Y)	39.1	39.4	-0.8	35.0	10.5
4	3 Vertical flexural (Y)	70.4	66.5	5.5	61.2	13.1
5	4 Vertical flexural (Y)	111.3	104.5	6.1	95.15	14.5

It can be noted that DYBEAM gives a mean difference of 2.5% in flexural vertical modes, meanwhile SAP2000 gives a mean difference of 7.3% for the same modes. In connection with lateral flexural modes, DYBEAM gives a mean difference of 9.5% and SAP2000 gives a mean difference of 13.2%.

c) Beam 3: Results are presented in Tables 6 and 7.

Table 6: Numerical-experimental comparison. Beam 3 clamped-free.

Frequency	Mode	Test Hz	DYBEAM[7,8] Freq. Hz	DYBEAM[7,8] Diff. %	SAP2000[20] Freq. Hz	SAP2000[20] Diff. %
1	1 Vertical flexural (Y)	8.8	8.8	0.0	8.8	0.0
2	1 Torsional	17.8	17.5	1.7	14.7	17.4
3	1 Vertical flexural (X)	-	39.8	-	39.5	-
4	2 Vertical flexural (Y)	45.8	44.8	2.2	42.0	8.3
5	2 Torsional	-	82.7	-	71.2	-
6	3 Vertical flexural (Y)	108.2	108.6	-0.6	96.6	10.7

Table 7: Numerical-experimental comparison. Beam 3 clamped-clamped.

Frequency	Mode	Test	DYBEAM[7,8]		SAP2000[20]	
		Hz	Freq. Hz	Diff. %	Freq. Hz	Diff. %
1	1 Vertical flexural (Y)	39.0	40.0	-2.6	36.2	7.2
2	1 Lateral flexural (X)	78.0	80.4	-3.1	71.0	9.0
3	2 Vertical flexural (Y)	104.4	104.5	-0.1	93.4	10.5

It can be noted that DYBEAM gives a mean difference of 1.1% in flexural vertical modes, meanwhile SAP2000 gives a mean difference of 7.3% for the same modes. In connection with lateral flexural modes, DYBEAM gives a mean difference of 2.4% and SAP2000 gives a mean difference of 13.2%.

d) Beam 4: Results are presented in Tables 8 and 9.

Table 8: Numerical-experimental comparison. Beam 4 clamped-free.

Frequency	Mode	Test	DYBEAM[7,8]		SAP2000[20]	
		Hz	Freq. Hz	Diff. %	Freq. Hz	Diff. %
1	1 Vertical flexural (Y)	6.2	6.3	-1.6	6.4	-3.2
2	1 Torsional	18.0	16.4	8.9	12.8	28.9
3	1 Lateral flexural (X)	-	28.1	-	28.5	-
4	2 Vertical flexural (Y)	29.2	32.0	-9.6	30.6	-3.3
5	3 Vertical flexural (Y)	59.2	62.9	-6.2	56.3	4.9
6	2 Torsional	-	75.7	-	63.9	-
7	4 Vertical flexural (Y)	116.8	113.6	2.7	103.5	11.4

Table 9: Numerical-experimental comparison. Beam 4 clamped-clamped.

Frequency	Mode	Test Hz	DYBEAM[7,8] Freq. Hz	DYBEAM[7,8] Diff. %	SAP2000[20] Freq. Hz	SAP2000[20] Diff. %
1	1 Vertical flexural (Y)	24.0	24.5	-2.1	23.3	2.9
2	2 Vertical flexural (Y)	59.0	58.8	0.3	55.6	5.8
3	1 Lateral flexural (X)	-	65.4	-	58.0	-
4	3 Vertical flexural (Y)	100.7	105.9	-5.2	101.1	-0.4

It can be noted that DYBEAM gives a mean difference of 4.0% in flexural vertical modes, meanwhile SAP2000 gives a mean difference of 4.6% for the same modes. In connection with lateral flexural modes, DYBEAM gives a mean difference of 8.9% and SAP2000 gives a mean difference of 28.9%.

5 CONCLUSIONS

In this paper, the equations of motion for thin-walled, variable open cross-section beams have been presented within the so-called state variables approach in the frequency domain. The equations take into account the influence of shear flexibility and rotatory inertias which are neglected in the original Vlasov's theory. The equations enable the analysis of practical problems using direct numerical integration in conjunction with techniques routinely applied in transfer matrix analyses. In addition, they may be resorted to in order to numerically evaluate transfer matrices or stiffness matrices for open section beam elements. Moreover, the proposed theory can also be used for solid beams in which coupling between bending and torsion occurs.

A numerical-experimental comparison was presented for natural frequencies of doubly unsymmetrical thin walled beams. This comparison is useful in order to obtain guides to the numerical modelling and analysis of this phenomenon. Moreover, the experimental could be used for checking the accuracy of a variety of calculation methods and theories developed for other authors.

Experimental studies show very good agreement between the results obtained with the theory presented in this paper and those obtained in the tests. Moreover, it is contended that the accuracy of this theory is comparable to shell FEM solutions. Finally is observed that, the theory presented in this paper maintain the accuracy for higher modes, which do not happen with the FEM model.

Acknowledgements

The author wishes to thank the collaboration of Profs. Jorge Riera and Rodolfo Danesi, the help received from technical staff, Mr. Eduardo Batalla and Mr. Daniel Torielli, during the development and preparation of the tests and Ms. Amelia Campos in the English revision. Moreover, the financial support of the CONICET and the Universidad Nacional de Tucumán is gratefully acknowledged.

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