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A RACE IN THE MONTE CARLO METHOD: NUMERICAL AND ANALYTICAL METHODS

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Abstract. Most real mechanical problems present nonlinear characteristics and are governed by nonlinear initial value problems (IVP). These nonlinear IVP usually do not present a known analytical solution. For this reason, approximation methods are employed. There exist many well-known analytical and numerical approximation techniques available in the literature which prove efficient in this assignment and can provide approximations up to any desired precision. While both methods succeed in presenting a approximation to IVP, the analytical methods have the advantage of deriving analytical approximation where the model parameters, initial conditions, etc, are explicitly present in the resultant mathematical expressions. This renders the latter a very cost-effective solution from a computational point of view, when we are interested in evaluating the model response in circumstances where those parameters or conditions change. Moreover, analytical approximations can very convenient when dealing with a stochastic approach, and especially with the Monte Carlo method. The Monte Carlo method is an important tool to build statistical models of random objects transformations. Usually, to construct an accurate statistical model (often histograms and sample statistics), several samples of the transformation output are required. If each sample is obtained by a numerical integration, the computation of the Monte Carlo method becomes a time-consuming task, with a high computational cost. An alternative to reducing those costs is to employ an analytical approximation. In this case, the computation of a numerical integration for each realization is replaced by a simple substitution in the expression of the analytical approximation, a much less costly task than that of the numerical methods. Therefore, the aim of this article is to quantify and compare the computational costs of the construction of statistical models by means of the Monte Carlo method using numerical and analytical approximations. The objective is to compare the gain in terms of CPU time when analytical approximations are used over numerical approximations. For this, a classic example of the literature will be used: an IVP involving the Duffing equation where one of the parameters of the problem will be modeled as a random variable.

1 INTRODUCTION

Dynamical problems governed by nonlinear initial value problems (IVP) ordinarily do not present a known analytical solution. In such case, an alternative is to compute approximate solutions, which also prove helpful in the understanding of the characteristics of their dynamics. Numerical and analytical methods are efficient in this assignment and can provide approximations up to any desired precision. Numerical methods have been widely applied in this type of problems. There exist many computational packages for their implementation, such the software MATLAB which includes several algorithms to perform numerical integrations. The most famous MATLAB 4^{th} and 5^{th} order Runge-Kutta method implementation is chosen for the numerical solutions in this paper. However, analytical approximations have an advantage in relation to numerical ones: they allow a deeper understanding into how the solution depends on the problem parameters. There exist many analytical techniques in the literature that allow us in this assignment (Sanchez (1996), Gomes (2019c), Gomes (2019d)). One of this method is the Lindstedt-Poincaré, a powerful technique to compute periodic analytical approximations. It transforms a nonlinear IVP, for which we do not know the analytical solution, into a linear-IVP family which we know to solve each one of them.

Parametric analysis is very important in the study of dynamic systems, as it helps to identify the phenomenon originated by the nonlinearity.This analysis, when carried out by means of a numerical approximation, has a high computational and temporal cost, since for each value of the parameter it is necessary to do a numeric integration Pasquetti (2008). The cost to do such analysis could be even higher if we take a stochastic approach as we can see in Lima (2015), Lima (2017a) and Lima (2017b). In the paper Lima (2017b) a parametric study of a stochastic nonlinear IVP was performed. The influence of two parameters was analyzed, one of them assumed 40 different values and the other 8, totalizing 320 combinations. As the article considers uncertainties and is aimed to building a statistical model, for each combination of the parameters it was necessary to do a Monte Carlo simulation. The Monte Carlo method is an important tool, which permits us to construct statistical models of random objects transformations. To obtain an accurate statistical model (often histograms and sample statistics), usually several samples of the transformation output are required. In each Monte Carlo simulation the IVP was be integrated 2,000 times, totaling 640,000 numerical integrations. To perform all of these calculations sequentially, 2,5 years would be required. Alternatively, the parallelization strategy was used. Integrations have been distributed into 16 computers, reducing the simulation time for 55 days. Note that even with parallelization, the computational cost remained high and took almost 2 months. An alternative to improve the computation time is to replace the numerical integrations by employing analytical approximations for the solution of each realization of the Monte Carlo Method. By doing this, instead of computing a numerical integration, we only need to do substitutions into the analytical expressions for each realization. A much less costly task. To verify this, we compute analytical and numerical approximations to the Duffing equation and chose as random variable the initial displacement.

2 ANALYTICAL APPROXIMATION

The Lindstedt-Poincaré method is an analytical technique to compute approximations to nonlinear initial value problems, which ordinarily do not have a analytical solution. In this context, these analytical methods are a helpful tool to study the response of those systems. A traditional example in the literature is the IVP involving the Duffing equation, which presents a

cubic nonlinearity, defined as

$$\ddot{x} + \omega_0^2\, x \,+\, \gamma\, x^3 \,=\, 0\,, \tag{1}$$

with initial conditions $x(0) = A_0$ and $\dot{x} = 0$. Where $\dot{\square}$ means the derivative with respect to t, γ is a constant parameter that controls the nonlinearity. Due to the cubic term, the Eq. (1) is nonlinear and an analytical solution is not known. As an alternative, we can use the Lindstedt-Poincaré method to compute an analytical approximation. This method transforms a nonlinear initial value problem in a linear-IVP family with known analytical solution for each linear IVP. The method considers as solution to the nonlinear IVP, a uniform expansion in relation to the perturbation parameter that should be introduced multiplying the nonlinear term (Gomes (2019a)). Replacing the expansion in the equation and doing some algebraic manipulations, we can transform the nonlinear IVP into a linear- IVP family. To find the original analytical solution we must calculate all linear IVP. However, as it is cumbersome to solve all these IVP, we usually consider N terms of the expansion. By doing this, we say that we have an approximation of order N. In Bender (1999), Radhika (2015), Gomes (2018), Gomes (2019b) more details about this methodology are provided. Therefore, we computed an analytical approximation of second order given by

$$\begin{aligned} x(t) \approx & A_0 \cos\omega t + \epsilon\,\gamma \frac{A_0^3}{32\,\omega_0^2}\left(\cos 3\omega t - \cos\omega t\right) \\ & + \epsilon^2\,\gamma^2\,\frac{A_0^5}{128\,\omega_0^4}\left(\frac{23\cos\omega t}{8} - \frac{3\cos 3\omega t}{1} + \frac{\cos 5\omega t}{8}\right) \end{aligned} \tag{2}$$

$$\omega(A_0) \approx \omega_0 + \epsilon\,\gamma\,\left(\frac{3\,A_0^2}{8\,\omega_0}\right) - \epsilon^2\,\gamma^2\,\left(\frac{21\,A_0^4}{256\,\omega_0^3}\right)\,. \tag{3}$$

3 STOCHASTIC APPROACH - MONTE CARLO METHOD

When we work with stochastic problems, the Monte Carlo method is an important tool due its ability to obtain numerical approximations for complex problems. It is used to make predictions about stochastic objects (Shonkwiler (2009)) and are grounded in two theorems, the law of large numbers and the central limit theorem (Sampaio (2012)). This method is commonly used in random object transformations when the transformation is difficult to compute, or when we do not know how to perform the analytical transformation of the probabilistic mode. So, we use the Monte Carlo method to generate a statistical model through a probabilistic one that we know how to build.

As random variable, we chose the initial displacement, A_0, and modeled it as a Beta function. Its probability density function (PDF) is

$$\frac{x^{\alpha-1}\,(x-1)^{\beta-1}}{B(\alpha,\beta)}\,, \tag{4}$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and Γ is the Gamma function with support $[0\,,200]$. We fixed the parameter $\beta = 2$ and chose 66 different values for α between $[1.5\,,8]$. For those values the shape of the PDF changes, as shown in Fig. 1. So, for values of α closer to 1.5 we have more probability to have values of A_0 under 100, while for values of α closer to 8 there is a higher probability of having values of A_0 above 100.

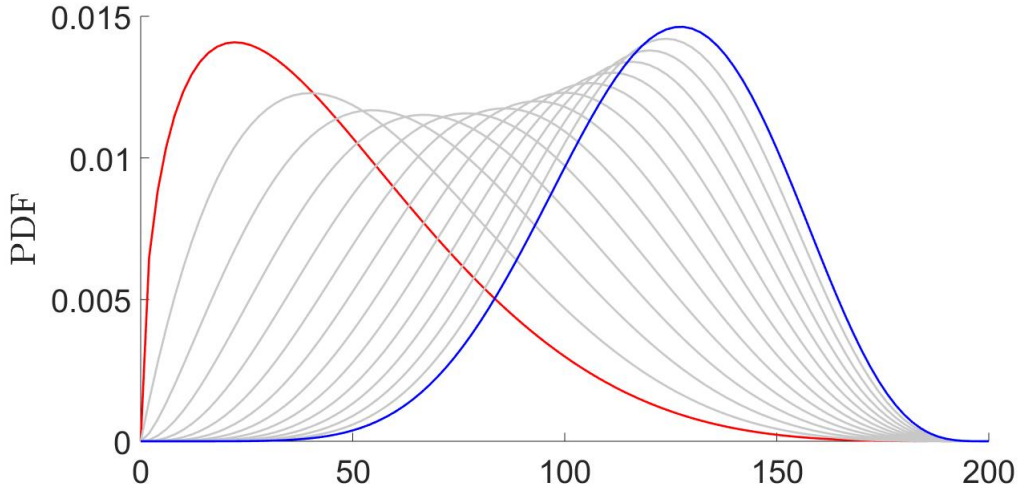

Figure 1: Probability density function of Beta distribution

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To analyze the temporal cost to perform a Monte Carlo simulation using analytical and numerical approximations, we vary some parameter values to see its influence in the CPU time. For the numerical approximation, we used the Runge-Kutta method of 4^{o} and 5^{o} order to integrated the Eq. (1), and to analytical approximation we used the Eq. (2). We considered 66 sample of Beta distribution for each value of α between $[1.5\,,8]$ with 10^4 realizations each one. Figure 2 shows the CPU time in seconds to the analytical and numerical approximations for each of the 66 samples of the Beta distribution. For this case, the numerical approximation took almost $5,66$ hours in the total, and the analytical approximation took almost 100 times less, only $3,2$ minutes.

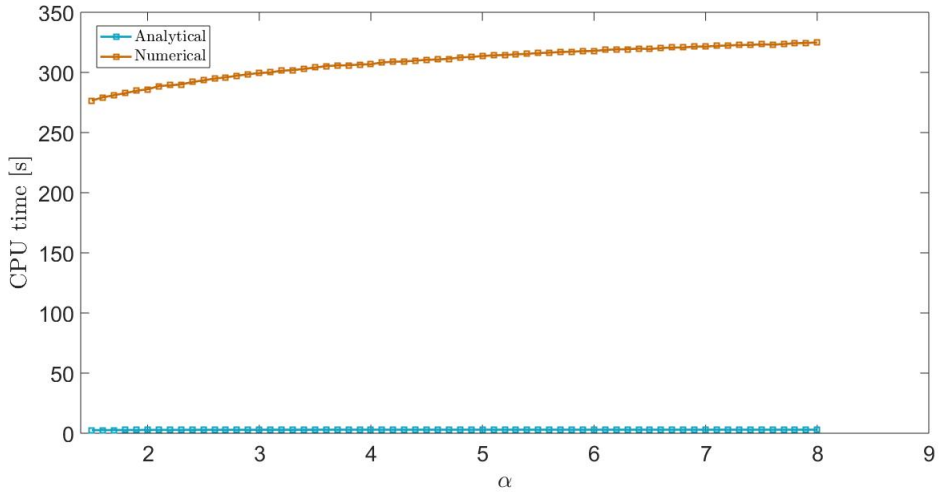

Figure 2: CPU time of the analytical and numerical distribution for each of the 66 samples of Beta distribution, considering $\gamma = 1.0$, $\epsilon = 1.0$.

We also analyze the influence of γ, since this parameter controls the nonlinearity of the system, higher its value, more nonlinear becomes the equation. So γ assumed 11 values between

$[0\,,2]$. Thereby, for each realization with a pair of values α and γ, we integrated the numerical approximation and evaluated the analytical approximation $7.26\,\text{x}10^6$ times.

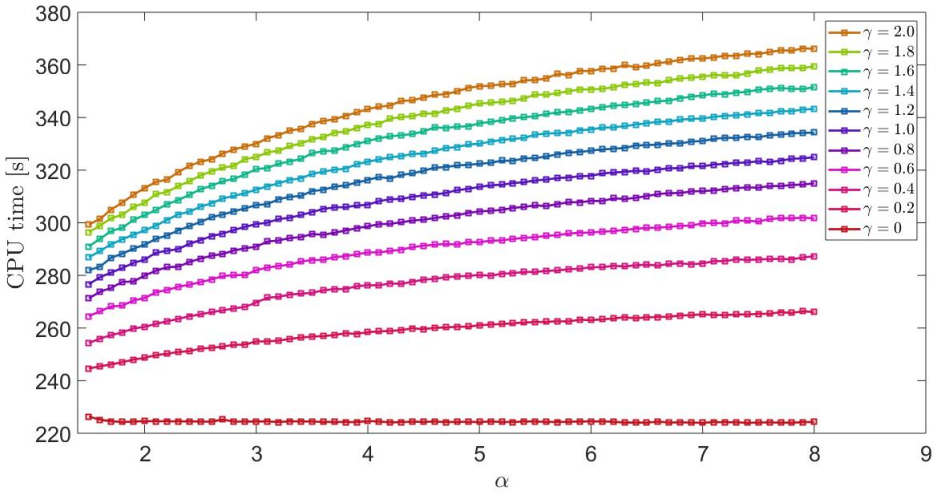

Figure 3: CPU time of the numerical distribution for each of the 66 samples of Beta distribution, considering $0 < \gamma < 2.0$, $\epsilon = 1.0$.

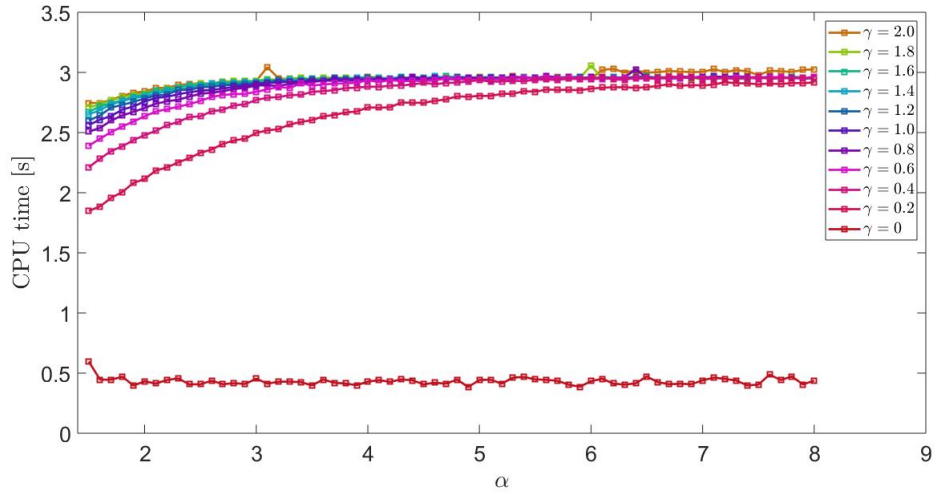

Figure 4: CPU time of the analytical distribution for each of the 66 samples of Beta distribution, considering $0 < \gamma < 2.0$, $\epsilon = 1.0$.

Figures 3 and 4 shows the CPU time to perform the numerical and analytical approximation, respectively. To compute the Monte Carlo method for all combinations of γ and α with the numerical approximation, it took almost $60,75$ hours, while with analytical approximation it took almost $32,16$ minutes. When γ assume value equal to 0 the Eq. (1) becomes linear, and in the both cases, numerical and analytical, the CPU time is much less than other curves with non-zero γ values. This cases, with nonzero γ, with the numerical approximation the CPU time increases as γ and α increase. While with analytical approximations the CPU times does not varies significantly.

5 CONCLUSIONS

With the simulations using analytical and numerical approximation to perform the Monte Carlo method, we can verify that the numerical integrations are very sensitive to the parameters values. The temporal cost to integrate the Eq. (1) using the Runge-Kutta method of 4^{o} and 5^{o} order is influenced by the value of γ and α, the Beta distribution parameters. It can also be observed that the CPU time to compute a numerical approximation is responsive to the values of the parameters, while the computational cost of the analytical approximation does not present a significant change. Moreover, observing the Figs. 3 and 4, it seems to us that the CPU time will continue to increase with the increase of the γ and α value, whereas for the analytical approximation the CPU time seems to continue in the same time range. In addition, there is an important difference in the magnitude of the time taken to calculate each of the approximations, while the numerical approximations takes hours to compute the analytical takes minutes.

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