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ELECTROMECHANICAL SYSTEM WITH A STOCHASTIC FRICTION FIELD

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Abstract.

This work analyzes the stochastic nonlinear dynamics of an electromechanical system with dry-friction. The system has a mechanical part and an electromagnetic part. It is composed of a cart, whose motion is excited by a DC motor. The coupling between the motor and the cart is made by a mechanism called scotch yoke, so that the motor rotational motion is transformed in horizontal cart motion over a rail. It is considered the existence of random dry-friction between the cart and the rail. The friction coefficient is modeled as a stochastic Poisson field over the rail. Due to the friction, the resulting motion of the motor can be characterized by two qualitatively different modes, the stick- and slip-modes, with a non-smooth transition between them. The focus of the work is finding, from a stochastic point of view, the stick- and slip-mode parts of the trajectory, i.e., to compute the uncertainty propagation in the system response.

1 INTRODUCTION

The analysis of electromechanical systems is not a new subject. The interest of analyzing their dynamic behavior is shown by the increasing amount of research in this area. Electromechanical systems present an interesting behavior characterized by the mutual influence between the electrical and mechanical subsystems of the system, that is, the dynamics of the motor is heavily influenced by the mechanical subsystem and the dynamics of the mechanical subsystem depends on the dynamics of the motor (Dantas et al., 2014, 2016; Clerkin and Sampaio, 2017; Manhaes et al., 2018) Each subsystem of the system affects the behavior of the other, i.e., they interact. The coupling varies with the coupling conditions, it is not a functional relation and depends on the initial conditions (Lima and Sampaio, 2016c, 2012) The dynamics of the coupled system is given by an initial value problem comprising a set of coupled differential equations (Lima et al., 2018; Lima and Sampaio, 2018).

The problem of coupled systems becomes even more interesting if it is considered the existence of dry-friction in the mechanical subsystem of the electromechanical system. The nonlinearity provided by the friction can induce stick-slip oscillations on the mechanical subsystem (Lima and Sampaio, 2015a, 2017a,b, 2013). Depending on the values of the system parameters, the response of the mechanical subsystem can be composed of a sequence alternating stick and slip-modes, (Lima and Sampaio, 2015b, 2016b, 2017d,c). The stick and slip-modes have a non-smooth transition (Ding, 2012; Fidlin, 2006; Galvanetto, 1999). The interest of analyzing the stick-slip dynamics is reflected by the increasing amount of research in this area (see for instance (Thomsen and Fidlin, 2003; Awrejcewicz and Olejnik, 2007; Hinrichs et al., 1998; Feng, 2003)).

The dry friction force presents an inherent random behavior (Feng, 2003). The influence of ambient conditions in the properties of contact surfaces (Bengisu and Akay, 1999; Worden et al., 2007) and the dependency on the relative velocity of the bodies in contact turn the dry friction force uncertain. Because of this, a stochastic approach is the ideal way to address the problem of dry-friction (Lima and Sampaio, 2014, 2015c, 2016a).

In this paper an electromechanical system with a stochastic friction field is analyzed. The works (Lima and Sampaio, 2019b,a) deal with the same system analyzed in this paper. However, in those works, the friction is considered to be deterministic. Here, the friction is random. The source of uncertainties is the friction coefficient between some the bodies in contact in the mechanical subsystem. It is modeled as a stochastic Poisson field.

The objective of this paper is to quantify the uncertainty propagation (Lima and Sampaio, 2018) in the response of the electromechanical system with stochastic dry-friction. Some of the variables of interest in the system response are the number of time intervals in which stick and slip occur, the instants at which they begin, and their durations. The system response presents a sequence of stick and slip-modes and we are interested in the uncertainty quantification of these sequences.

2 DYNAMICS OF THE ELECTROMECHANICAL SYSTEM WITH DRY-FRICTION

The system analyzed in this paper is composed by a cart-disk whose motion is driven by a DC motor. The motor is coupled to the cart through a pin that slides into a slot machined on a plexiglas plate that is part of the cart, as shown in Fig. 1. The pin hole is drilled off-center on a disk fixed in the axis of the motor, so that the motor rotational motion is transformed into horizontal cart motion over a rail. The eccentricity influences heavily the nonlinearity of the system. Even small eccentricities produces high nonlinearities. The dynamics of a DC motor is

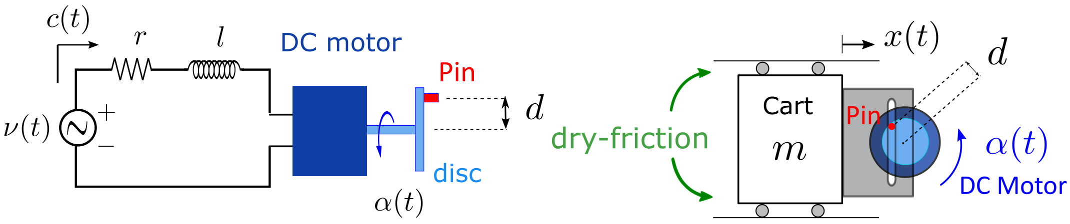


Figure 1: Electromechanical system with dry-friction between the cart and the rail.

given by the following initial value problem (IVP). Given a source voltage ν, find $(\alpha,\, c)$ such that, for all $t > 0$,

$$l\dot{c}(t) + r\; c(t) + k_e\dot{\alpha}(t) = \nu(t)\;, \tag{1}$$

$$j_m\ddot{\alpha}(t) + b_m\dot{\alpha}(t) - k_e\; c(t) = -\tau(t)\;, \tag{2}$$

with the initial conditions

$$\dot{\alpha}(0) = \dot{\alpha}_0 \quad , \quad \alpha(0) = \alpha_0 \quad , \quad c(0) = c_0\,, \tag{3}$$

where t is the time, ν is the source voltage, c is the electric current, $\dot{\alpha}$ is the angular speed of the motor, l is the electric inductance, j_m is the motor moment of inertia, b_m is the damping ratio in the transmission of the torque generated by the motor to drive the coupled mechanical system, k_e is the motor electromagnetic force constant and r is the electrical resistance. The modulus of the available torque to the coupled mechanical system is τ. The source voltage is considered to be

$$\nu(t) = \nu_0 + \nu_1 \sin\left(\omega_v\, t\right). \tag{4}$$

The mass of the mechanical system is m and the horizontal cart displacement is represented by x. It is considered that the cart is not allowed to move in the vertical direction. Due to the problem geometry, and noting $||\boldsymbol{d}|| = d$, the horizontal motion of the cart and the angular displacement α of the motor are related by the constraint

$$x(t) = d\, \cos\left(\alpha(t)\right). \tag{5}$$

In the model of the coupling between the motor and the mechanical system, it is assumed that the motor shaft is rigid. Thus, the available torque to the coupled mechanical system, $\boldsymbol{\tau}$, can be written as

$$\boldsymbol{\tau}(t) = \boldsymbol{d}(t) \times \boldsymbol{f}(t)\,, \tag{6}$$

where $\boldsymbol{d}$ is the eccentricity of the pin of the motor, considered a vector, and $\boldsymbol{f}$ is the coupling force between the DC motor and the cart. The component of $\boldsymbol{d}$, which is perpendicular to the plane of the cart motion, is always zero, and the others horizontal and vertical components can be calculated from the angular displacement α of the disk. Assuming that there is no friction between the pin and the slot machined on an acrylic plate, the vector $\boldsymbol{f}$ only has a horizontal component, called f, which is the horizontal force that the DC motor exerts in the cart. Thus, the modulus of $\boldsymbol{\tau}(t)$ is

$$\tau(t) = -f(t) d \sin\alpha(t)\,. \tag{7}$$

Since the cart is modeled as a particle, its motion in the horizontal direction satisfies the equation:

$$m\,\ddot{x}(t) = f(t) + f_r(t)\,, \tag{8}$$

where f_r is the dry-friction force between the cart and the rail. The initial value problem to the coupled motor-disk-cart system with dry-friction is: given v, find $(\alpha,\ c)$ satisfying

$$\begin{aligned} l\dot{c}(t) \quad & + r\ c(t) + k_e\dot{\alpha}(t) = \nu_0 + \nu_1 \sin(\omega_v\, t)\,, \\ \ddot{\alpha}(t) \quad & \left[j_m + m\, d^2 (\sin\alpha(t))^2\right] - k_e\ c(t) + \\ & + \dot{\alpha}\ \left[b_m + m\, d^2 \dot{\alpha}(t) \cos\alpha(t)\ \sin\alpha(t)\right] = -f_r(t)\, d\, \sin\alpha(t)\,, \end{aligned} \tag{9}$$

for given initial conditions of electric current, angular velocity and position of the motor. The friction is modeled as Coulomb's, which is a simple model, shown in Fig. 2. This model was chosen because of its simplicity. Different friction models, with hysteresis loops, for example, may leads to different results. The study of different friction models is an interesting research topic and will be investigated in future works.

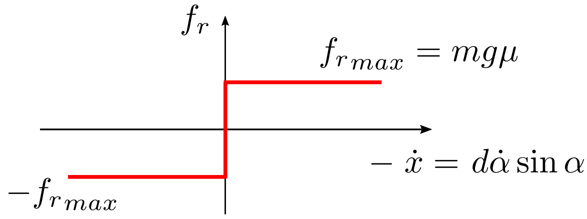


Figure 2: Coulomb dry-friction.

The non-smooth behavior of the dry-friction force can induce in the system stick-slip oscillations. Depending on the values of the system parameters, the response of the system may be composed of a sequence alternating stick and slip-modes. In Fig. 3 it is shown a possible sequence of stick and slip-modes for the interval of analysis $[0, t_a]$.

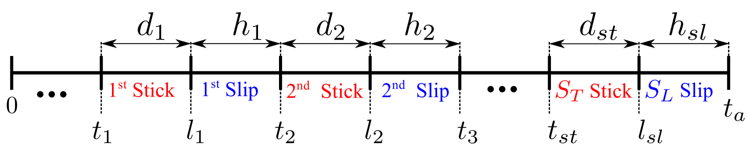


Figure 3: Sequence of sticks and slips in a system response.

During the stick-mode, the disk-cart does not move, so that the angle, describing the angular position of the disk, is constant. The frictional force and the current, however, can vary. There is an internal dynamics of the system in the electromagnetic subsystem. Hence, stick means only no motion of the disk-cart, the mechanical subsystem. The electromagnetic subsystem continues to change its state until it gathers enough power to move the disk-cart again. The stick mode occurs when $\dot{\alpha} = 0$ in an interval and when the frictional force, which satisfies

$$k_e\ c(t) = f_r(t)\, d\ \sin(\alpha(t))\,, \tag{10}$$

is in the interval $-f_{max} \leq f_r \leq f_{max}$, where $f_{max} = \mu m\, g$, g is the gravity and μ is the friction coefficient between the cart and the rail. Equation (10) is obtained considering $\dot{\alpha} = 0$ and $\ddot{\alpha} = 0$ in Eq. (9). Remark that during the stick-mode, the frictional force varies and depends on the angular position of the motor. There is a functional relation between these two variables. Besides, the initial value problem that describes the dynamics of the coupled motor-cart system with dry-friction is reduced to just one differential equation given by

$$l\dot{c} + r\, c = v_0 + v_1 \sin(\omega_v\, t)\,, \tag{11}$$

where the initial condition for the current is the value of it in the beginning of the stick-mode. Observe that during the stick-mode, the sum of the forces that act over the cart is zero (it does not move). The horizontal coupling force between the DC motor and the cart, f, is balanced by the dry-friction force, f_r. This balance lasts until the frictional force, given in Eq. (10), reaches its maximum value, f_{max}. During the stick-mode, the dynamics of the system is governed only by the dynamics of the electrical circuit of the motor. Electrical and mechanical subsystems do not interact during the duration of a stick, the subsystem are decoupled. During the slip-mode, the dry-friction force is

$$f_r(t) = -m\, g\mu\, \mathrm{sgn}(\dot{x}(t)) = -m\, g\mu\, \mathrm{sgn}(-\dot{\alpha}(t)\, d\, \sin(\alpha(t)))\,. \tag{12}$$

3 PROBABILISTIC MODEL TO THE FRICTION COEFFICIENT

We model the friction coefficient as a stochastic Poisson field between the cart and the rail (Cox and Isham, 1980; Kingman, 2002). A realization of such stochastic field consists of point events in the interval $[-d,\ d]$ which represents the positions in which occur changes of the friction coefficient. Each realization is constant by parts and assumes only two values: μ_1 and μ_2. Figure 4 shows a sketch of a realization of the friction coefficient field.

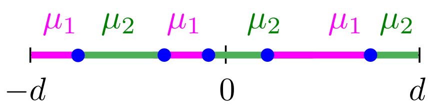


Figure 4: Sketch of a realization of the friction coefficient field.

The number of changes of the friction coefficient in the interval $[-d,\ d]$ is given by a random variable N with Poisson distribution with parameter $\lambda 2\, d$. Thus, for $n = 0, 1, 2, \ldots$, the probability mass function of N is given by

$$\Pr(N = n) = \frac{(\lambda 2\, d)^n\, e^{-\lambda 2\, d}}{n!}\,, \tag{13}$$

Observe that λ represents the expected value of number of changes per unit of length. Thus, given the interval $[-d,\ d]$, the number of changes of the friction coefficient has a Poisson distribution, with a parameter which is proportional to the length of the interval. Note that, so far, nothing was said about the positions in which the changes occur. To determine them, we use the fact that we want our random process to be stationary and to be the most uncertain as possible. Therefore, it is reasonable to distribute all positions of changes over the interval $[-d\,, d]$ in a completely arbitrary way. Then, these instants of changes are modeled as independent and identically distributed random variables, U_1, $U_2, \cdots, U_n$, each of them uniformly distributed over $[-d\,, d]$. Given a realization of U_1, $U_2, \cdots, U_n$, i.e. given the n-uple

$(u_1,\, u_2, \cdots, u_n)$, to generate a realization of the stochastic friction field on $[-d\,, d]$, we need to sort the samples. We transform the n-uple $(u_1,\, u_2, \cdots, u_n)$ in $(y_1,\, y_2, \cdots, y_q)$ in a way that $y_1 \leq y_2 \leq \cdots \leq y_n$. This operation generates new random variables, $Y_1,\, Y_2, \cdots, Y_n$, wherein $Y_1 = \min_{1 \leq i \leq n} \{U_1, \cdots, U_n\}$.

The following three-step procedure is used to generate a realization of $\mathcal{V}$ on $[0\,, t_a]$:

1. draw a sample, n, of N with Poisson distribution with parameter $\lambda 2\, d$. This number represents the total number of changes on $[0\,, t_a]$.

2. draw one sample of $U_1,\, U_2, \cdots, U_n$ each with uniform distribution over $[-d\,, d]$. This generates a n-uple $(u_1,\, u_2, \cdots, u_n)$ which represent the positions of change of the friction coefficient.

3. sort the the n-uple $(u_1,\, u_2, \cdots, u_n)$, i.e. the positions of change, in ascending order, to get $(y_1,\, y_2, \cdots, y_n)$.

4 CONSTRUCTION OF A STATISTICAL MODEL OF THE STICK-SLIP PROCESS

As it was assumed that the friction coefficient is a stochastic field, the response of the stochastic stick-slip oscillator is a random process which presents a sequence of stick and slip-modes. We are interested in the stochastic characterization of these sequences. Defined a time interval for analysis, some of the variables of interest are the number of time intervals in which stick and slip occur, the instants at which they start and their duration. These variables are modeled as stochastic objects in order to allow the stochastic characterization the dynamics of the oscillator.

- The number of time intervals in which stick occurs is represented by the discrete random variable S_T.

- The number of time intervals in which slip occurs is represented by the discrete random variable S_L.

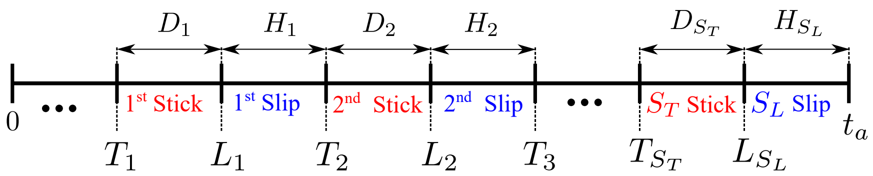
- The instants at which sticks begin are represented by a discrete random process $T_1, \cdots, T_{S_T}$, where the subscripts $1, \cdots, S_T$ indicate the order that they occur. i.e., the instant in which starts the first stick, the second, and so on up to the S_T-th stick;

- The duration of sticks is represented by a discrete random process $D_1, \cdots, D_{S_T}$, where again the subscripts $1, \cdots, S_T$ indicate the order that they occur.

- The instants at which slips begin are represented by a discrete random process $L_1, \cdots, L_{S_L}$, where $1, \cdots, S_L$ indicate the order that they occur.

- The duration of sticks is represented by a discrete random process $H_1, \cdots, H_{S_L}$, where $1, \cdots, S_L$ indicate the order that they occur.

Figure 4 shows a sketch of the sequence of sticks and slips in the system response. Observe that we count the first slip just after the first stick, i.e., we have $L_1 > T_1$. Besides this, if the chronometer stops during a slip, the number of sticks is equal or the number of slips, i.e. $S_T = S_L$. If the the chronometer stops during a stick, then $S_T = S_L + 1$.



5 UNCERTAINTY PROPAGATION IN THE SYSTEM RESPONSE

To estimate histograms of the random variables that characterize the system response, the dynamical equations were integrated 20,000 times using independent realizations of the base movement generated with the Monte Carlo method (Sampaio and Lima, 2012; Souza de Cursi and Sampaio, 2015). A previous convergence study was developed to determine the acceptable number of realizations. A large number of realizations of the system response was necessary in order to construct histograms with accuracy. For computation, duration t_a was chosen as 10 seconds and $\lambda = 10.0$ 1/m. For the integration, it was used the function *ode45* of the *Matlab* software, which applies the Runge-Kutta 4th/5th-order method as time-integration scheme with a varying time-step algorithm. The maximal step size is equal to 10^{-4} seconds, and the relative and absolute tolerance are equal to 10^{-9}. The values of the parameters used in all simulations are given in Table 1. The initial conditions of the system are $\alpha(0) = \pi$ [rad], $\dot{alpha}(0) = 0$ [rad/s], and $c(0) = \nu_0/r$.

$l = 1.880 \times 10^{-4}$ H	$k_e = 5.330 \times 10^{-2}$ volts/(rad/s)
$j_m = 1.210 \times 10^{-4}$ kg m^2	$r = 0.307\ \Omega$
$b_m = 1.545 \times 10^{-4}$ Nm/(rad/s)	$m = 5.000$ kg
$\nu_0 = 1.000$ volts	$\nu_1 = 0.500$ volts
$\omega_v = 10.000$ rad/s	$d = 0.010$ m
$\mu_1 = 0.300$	$\mu_2 = 0.500$

Table 1: Parameter values

Figure 5 shows the histogram of the number of changes of the friction coefficient, i.e., the random variable N with Poisson probability mass function. Figure 6 shows the normalized histograms of the first six positions of discontinuities of the friction coefficient, i.e., random variables $Y_1, \cdots, Y_6$. Figure 7 and 8 show the normalized histograms of the first six instants at

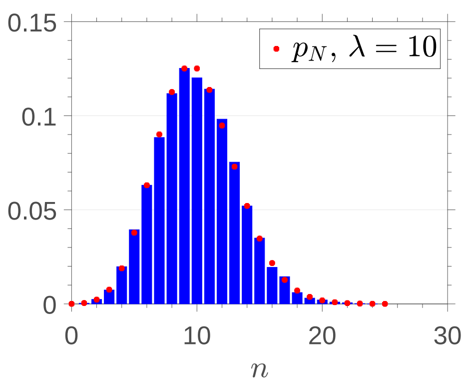


Figure 5: Normalized histogram constructed with 20,000 samples of the number of discontinuities of the friction coefficient.

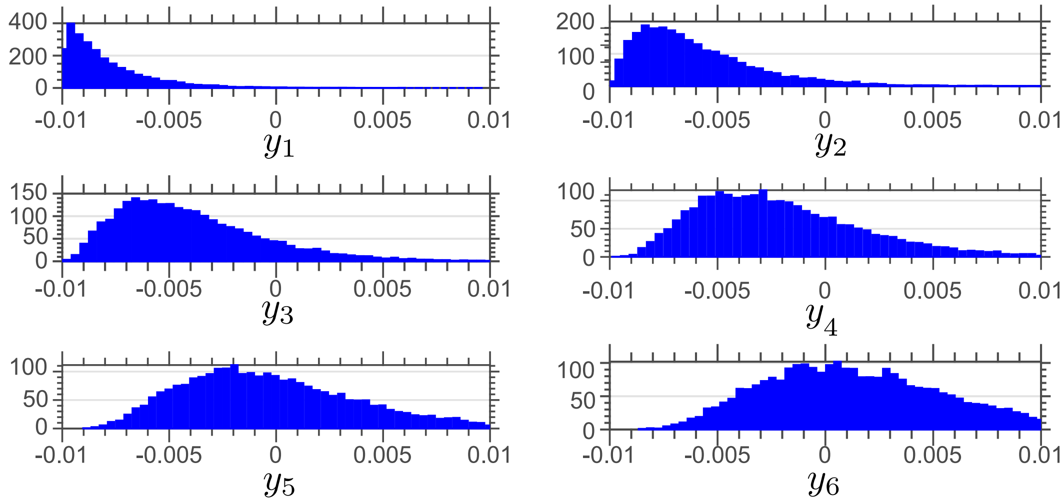


Figure 6: Normalized histogram constructed with 20,000 samples of the first six positions of discontinuities of the friction coefficient, $Y_1, \cdots, Y_6$.

which the sticks begin, (T1, $\cdots$,T6) and the normalized histograms of the duration of the first six sticks $(D_1, \ldots, D_6)$. The normalized histogram of the angle of the disk in which the sticks

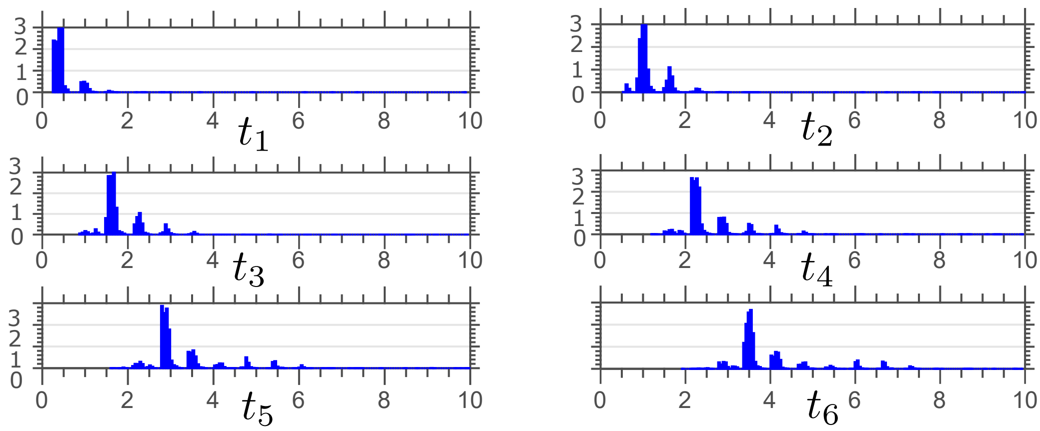


Figure 7: Normalized histogram constructed with 20,000 samples of the of the first six instants at which the sticks begin, (T1, $\cdots$,T6).

occur is presented in Fig. 9.

6 CONCLUSIONS

In this article it is analyzed the dynamics of an electromechanical system composed by a cart and a DC motor. The coupling between the motor and the cart is made by a mechanism called scotch yoke, so that the motor rotational motion is transformed in horizontal cart motion on a rail. There is dry-friction between the cart and the rail. The resulting motion of the cart can be characterized by stick- and slip-modes, with a non-smooth transition between them. The friction coefficient is modeled as a random field in a way that the system response becomes a random sequence of stick- and slip-modes. The uncertainty propagation in the system response is quantified by histograms of the random variables that characterizes this random sequence of stick- and slip-modes.

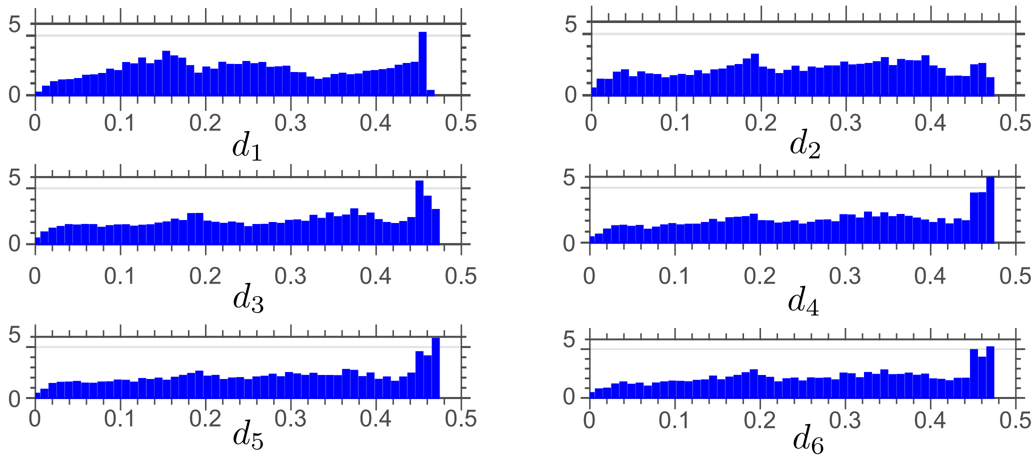


Figure 8: Normalized histogram constructed with 20,000 samples of the duration of the first six sticks ($D_1, \ldots, D_6$).

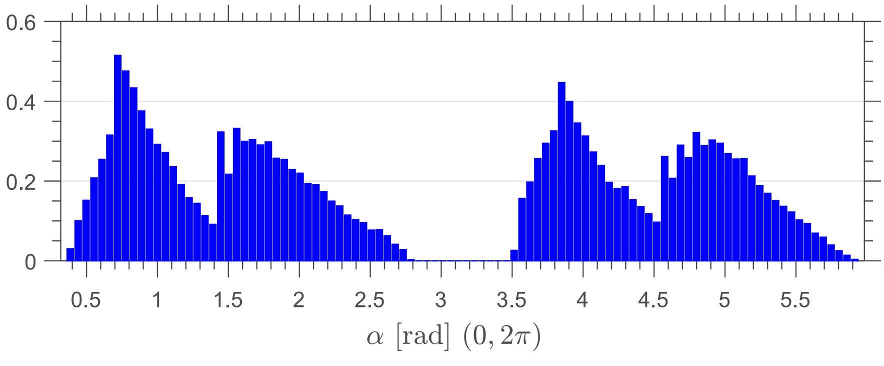


Figure 9: Normalized histogram constructed with 20,000 samples of the of the angle of the disk in which the sticks occur.

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