Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXVII, págs. 735-745 (artículo completo)
A. Cardona, L. Garelli, J.M. Gimenez, P.A. Kler, S. Márquez Damián, M.A. Storti (Eds.)
Santa Fe, 5-7 Noviembre 2019

DERIVATION OF DIFFERENT CONSTITUTIVE LAWS FOR THE COSSERAT MEDIUM

Héctor E. Goicoechea[a,b], Roberta Lima[d], Rubens Sampaio[d], Fernando S. Buezas[a,c] and Marta B. Rosales[a,b]

[a]*IFISUR, Universidad Nacional del Sur, CONICET, Departamento de Física - UNS, Av. L. N. Alem 1253, B8000CPB - Bahía Blanca, Argentina*

[b]*Departamento de Ingeniería, Universidad Nacional del Sur, Av. Alem 1253, (8000) Bahía Blanca, Argentina*

[c]*Departamento de Física, Universidad Nacional del Sur, Av. Alem 1253, (8000) Bahía Blanca, Argentina*

[d]*Dto. Engenharia Mecânica, PUC-Rio, Rua Marquês de São Vicente 225, 22453-900 Rio de Janeiro, Brasil*

Keywords: Coupled Electromechanical Systems, Piezoelectric Coupling, Coupled Cosserat Rod

Abstract.

In recent years, disciplines such as transport, space, and civil engineering are using lighter structural members to withstand the action of forces. On the one hand, this characteristic of structures results advantageous as smaller cross-sections are required, and therefore less material is used. On the other hand, the reduction of self-weight added into the design load-state makes these elements prone to suffering from strong undesired vibrations. Many different methods are addressed in the literature to mitigate the effect of unwanted vibrations: the installation of mass-tuned-dampers, the addition of viscoelastic materiales in the contact regions, friction dampers, or piezoelectric devices, among others. In this work the coupling mechanism for a passive vibration system in a beam-like structure modelled via the special theory of Cosserat rods is studied. The addition of a piezoelectric device in the mechanical structure is considered as a means to reducing the unwanted vibration phenomena and leads to a coupled electromechanical system. A procedure to derive the constitutive laws required for rod elements with mixed elastic material and piezoelectric devices is herein discussed. The present methodology could also be used to explore constitutive laws for different piezoelectric configurations, which could be of interest to control unwanted torsional vibrations in rotating structures such as drill-strings.

1 INTRODUCTION

Piezoelectric materials are widely used for many different applications in the area of vibration, measurement and control. The use of piezoelectric components can help achieve a desired response of a structural member, as they can be used both as means to monitoring and controlling flexible structures (Rao and Sunar, 2009). Many recent applications are also focused in energy harvesting by means of piezoelectric devices such as MEMS, that capture ambient vibrations (Erturk and Inman, 2011). Another interesting area of application is mechanical vibrations, where these materials are being added to regular beam-like structures in order to control unwanted vibrations, as shown in Ducarne (2009).

Two basic phenomena are involved in the behaviour of piezoelectric materials which allows them to act as both sensors and actuators in a control system. The first one, the so-called direct piezoelectric effect, implies that when a priezoelectric material is mechanically strained, electric polarisation that is proportional to the applied strain is induced. Therefore, some charge (or voltage) is induced under the application of a mechanical pressure. Conversely, the inverse effect (sometimes also called converse or reverse effect) implies that some imposed charge or voltage will provoke a reaction generating a mechanical strain (Erturk and Inman, 2011).

Piezoelectric components can be used in regular structures to reduce vibrations, and particularly in conjunction with passive electrical circuits so as to obtain the same efficiency as active vibration control, without the associated complexity and energy consumption (Ducarne, 2009).

Most of the applications of piezoelectric devices are limited to small strain and small displacement conditions. The objective of this work is to describe the necessary foundations to modelling piezoelectric devices in beam-like structures that can undergo large displacements, such as the Cosserat rod theory. The main idea is to obtain the necessary equations to deal with coupled electromechanical problems for this kind of structures. In a future work, it would be interesting to investigate the effect of different piezoelectric configurations in the control of axial, transverse and torsional vibrations, for which the constitutive relations can be derived in an analogous manner. For the time being, only the case of a piezoelectric devise polarised in the longitudinal axis will be analysed. Therefore the formulation provided can be used in the analysis of axial-transverse vibrations. The present formulation will be derived from the known 3D linear constitutive relations by imposing a set of hypothesis on the kinematic displacements.

A rod-like structure formed of layers of either elastic material or piezoelectric components is considered. For this reason, the constitutive expressions for both parts are described.

2 CONTINUUM MECHANICS NOTATION AND MATERIAL FORM OF ELECTRIC EQUATIONS

In the present work, the constitutive relations for rods are derived parting from known 3D constitutive expressions for linear piezoelectric materials. The objective is to obtain the relationship between stress, strain, electric field, and electric displacement in a Cosserat rod medium, considering large displacements and small strains. For this task it will be necessary to introduce some notation from continuum mechanics. The book by Gurtin et al. (2009) provides a detailed demonstration and definition of each of the magnitudes employed in the present work.

Let B be a body in the reference configuration (which is arbitrary) in the euclidean space $\mathscr{E}$, then the set of points that the body occupies are called material points and described by vector $\mathbf{X}$.

A motion of B is a smooth function χ that assigns a point to each material point at a given time. The current point (current configuration) can then be described as $\mathbf{x} = \chi(\mathbf{X}, t)$ in the current space (or observed space). With this notation, the deformation gradient $\mathbf{F}$ and its determinant J are defined as follows

$$\mathbf{F} = \nabla \mathbf{x}, \quad (\mathbf{F})_{ij} = \frac{\partial \mathbf{x}_i}{\partial \mathbf{X}_j} \quad J = det(\mathbf{F}) \tag{1}$$

Then, the following relations between the Cauchy $\mathbf{T}$, First Piola $\mathbf{T}^R$ and Second Piola $\mathbf{T}^{RR}$ stress tensors exist.

$$\mathbf{T}^R = \mathbf{F}\mathbf{T}^{RR}, \quad \mathbf{T} = J^{-1}\mathbf{F}^T\mathbf{T}^R \tag{2}$$

Next, the right Cauchy-Green tensor is defined in terms of the deformation gradient as $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and the Green-St.Venant strain tensor is calculated as $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, where $\mathbf{I}$ is the identity matrix. In the previous notation, no superscript $(\cdot)$ is used to refer to pure spacial objects, and the superscripts $(\cdot)^R$ and $(\cdot)^{RR}$ are used to refer to mixed objects and pure material objects, respectively. The notation $(\cdot)^{RR}$ is reserved to pullbacks of objects that originally belonged to the current space, while the superscript $(\cdot)^o$ is reserved to objects that belong to the material space.

Later on, the constitutive relation for the piezoelectric layers will be expressed in its material form. Following (Yang, 2005), the electric displacement $\boldsymbol{\mathcal{D}}$ and the electric field $\boldsymbol{\mathcal{E}}$ are transformed from pure material to mixed and spacial fields, as expressed next.

$$\boldsymbol{\mathcal{D}}^R = \mathbf{F}\boldsymbol{\mathcal{D}}^{RR}, \quad \boldsymbol{\mathcal{D}}^R = JF^{-1}\boldsymbol{\mathcal{D}} \tag{3}$$

$$\boldsymbol{\mathcal{E}}^R = \mathbf{F}\boldsymbol{\mathcal{E}}^{RR}, \quad \boldsymbol{\mathcal{E}}^R = JF^{-1}\boldsymbol{\mathcal{E}} \tag{4}$$

3 CONSTITUTIVE LAW FOR A COSSERAT MEDIUM WITH ELASTIC MATERIALS IN CROSS-SECTIONS WITH DOUBLE SYMMETRY

In this section, a procedure to deduce a constitutive law for an elastic material in rod theories following Linn et al. (2013) and Géradin and Cardona (2001) is presented. The law is derived parting from a known 3-D constitutive relation for for hyper-elastic material, i.e. where the stress-strain relation can be stated in terms of a strain energy density function. For such case, the constitutive law can be stated in terms of the second Piola tensor $\mathbf{T}^{RR}$, the Green-St. Venant tensor $\mathbf{E}$, and the Lamé constants μ and λ (Gurtin et al., 2009).

$$\mathbf{T}^{RR} = 2\mu\, \mathbf{E} + \lambda\, \mathrm{tr}(\mathbf{E})\, \mathbf{I} \tag{5}$$

The flowchart presented in Fig. 1 shows the steps involved in the derivation procedure.

3.1 Kinematic assumptions

The first step to deriving a constitutive model for rods is to introduce some kinematic assumptions for the kinematics of the body. Those assumptions will help simplify the expressions involved in the dynamics, and they are the key to describing the behaviour of a 3D body by means of a 1D domain. A common hypothesis employed in many beam theories is adopted: points that lie in the cross-section behave as rigid bodies. Therefore, cross-sections do not change shape but only orientation. Then, the kinematics of the body can be described by defining the position of the centreline and the orientation of the cross-sections.

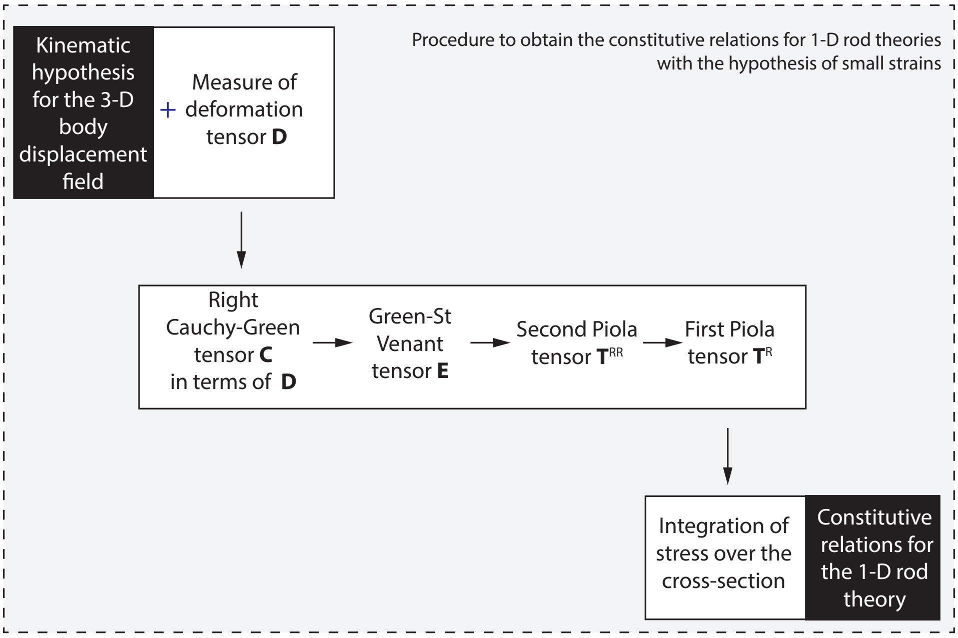

Figure 1: Description of the required steps to find a constitutive relation for 1D rod type theories.

A sketch of the rod at different times in the current configuration is presented below. For the current derivation, it is considered that the reference local frame coincide with the fixed frame, and that the reference configuration coincides with the current configuration at time $t = 0$ s, as shown in Fig 2.

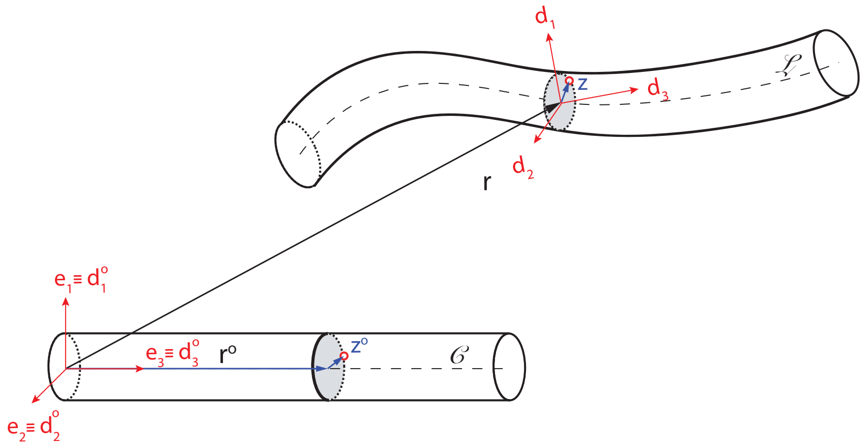

Figure 2: Sketch of the rod at different times, with the correspoding current local frames, and reference local and fixed frames.

Next, the displacement field for the 3-D body is stated in terms of the generalised coordinates that will be adopted for the rod theory.

In accordance to Fig. 2, let $\mathscr{C}$ and $\mathscr{L}$ be the centreline curves in the reference and current space. The unit vectors $\mathbf{d}_i^o$, $\mathbf{d}_i$ define a base in the reference and current space respectively. These are local bases that change orientation at each point of the centreline, alike the

Frenet-Serret frame employed in differential geometry. Also, let e_i define an inertial base in the reference configuration.

In the reference configuration, the position of any point can be described as

$$\mathbf{X} \quad = \quad \mathbf{r}^o(s^o) + \mathbf{z}^o(\zeta_1, \zeta_2) = \zeta_1 \mathbf{e}_1 + \zeta_2 \mathbf{e}_2 + s^o \mathbf{e}_3 \tag{6}$$

In the previous expression, s^o, ζ_1, ζ_2 are the components of the position vector of any point on the reference configuration, expressed in the inertial frame $\mathbf{e}_i$.

Next, let $\mathbf{Q}(s^o)$ be a rotation matrix. Due to the kinematic assumptions, if the cross-sections behave as rigid bodies, then the current configuration can be described in terms of the current centreline position and the rotated coordinates of the points that lay on the associated cross-section.

$$\mathbf{x} = \mathbf{r}(s^o) + \mathbf{Q}(s^o)\mathbf{z}^0(\zeta 1, \zeta 2) \tag{7}$$

3.2 The displacement gradient measure of deformation tensor

The displacement gradient measure of deformation tensor $\mathbf{D}$ introduced in Géradin and Cardona (2001), establishes a comparison in the material space between the position gradient before and after the deformation. This tensor provides a useful way to express the deformation gradient, and to introduce the hypothesis of small-strain.

$$\mathbf{D} \quad = \quad \mathbf{Q}^T \frac{\partial \mathbf{x}}{\partial \mathbf{X}} - \frac{\partial \mathbf{X}}{\partial \mathbf{X}} = \mathbf{Q}^T \mathbf{F} - \mathbf{I} \tag{8}$$

For the given kinematic assumptions, the tensor $\mathbf{D}$ is expressed as

$$\mathbf{D} = \begin{bmatrix} | & | & | \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & (\mathbf{D})_{31} \\ 0 & 0 & (\mathbf{D})_{32} \\ 0 & 0 & (\mathbf{D})_{33} \end{bmatrix} \tag{9}$$

with

$$\mathbf{D}_3 \quad = \quad \mathbf{Q}^T \frac{\partial \mathbf{x}}{\partial s^o} - \frac{\partial \mathbf{X}}{\partial s^o} = \mathbf{Q}^T \left(\frac{\partial \mathbf{r}^o}{\partial s^o} + \frac{\partial \mathbf{Q}}{\partial s^o} \mathbf{z}^o \right) - \frac{\partial \mathbf{X}}{\partial s^o} \tag{10}$$

As presented in (Cao and Tucker, 2008; Goicoechea et al., 2019), the variation of the directors within the arc-length can be stated in the following form

$$\frac{\partial \mathbf{d}_i}{\partial s^o} = \mathbf{u} \times \mathbf{d}_i, \quad \frac{\partial \mathbf{Q}}{\partial s^o} = \mathbf{u} \times \mathbf{Q} \tag{11}$$

It is then shown that the previous tensor $\mathbf{Q}^T(\mathbf{u}_\times)\mathbf{Q}$ is also skew-symmetric and can be written in the form of $(\tilde{\mathbf{u}}\times)$. The relation between $\mathbf{u}$ and $\tilde{\mathbf{u}}$ is stated next.

$$(\mathbf{Q}^T\mathbf{u})_\times = \mathbf{Q}^T(\mathbf{u})_\times\mathbf{Q} = (\tilde{\mathbf{u}})_\times \tag{12}$$

Introducing the notation $\mathbf{v}^R = \dfrac{\partial \mathbf{r}(s^o)}{\partial s^o}$ and $\mathbf{v}^{RR} = \mathbf{Q}^T\mathbf{v}^R$, $\mathbf{D}_3$ is written as follows

$$\mathbf{D}_3 \quad = \quad \mathbf{v}^{RR} - \mathbf{e}_3 + (\tilde{\mathbf{u}}_\times)\, \mathbf{z}^o \tag{13}$$

Next, from the previous definition for **D**, it is possible to find an expression for the deformation gradient **F**.

$$\mathbf{F} = \nabla \mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{Q}\Big(\mathbf{D} + \mathbf{I}\Big) \tag{14}$$

3.3 Right Cauchy-Green tensor

The right Cauchy-Green tensor is calculated in what follows, by employing expression (14).

$$\mathbf{C} \;\; = \;\; \mathbf{F}^T\mathbf{F} = \big(\mathbf{D}^T + \mathbf{I}\big)\mathbf{Q}^T\mathbf{Q}\big(\mathbf{D} + \mathbf{I}\big) = \mathbf{D} + \mathbf{D}^T + \mathbf{D}^T\mathbf{D} + \mathbf{I} \tag{15}$$

A small deformation hypothesis implies that $||\mathbf{D}||_{\mathbb{F}} \to 0$, where $||\cdot||_{\mathbb{F}} = \sqrt{D:D} = \sqrt{D^T D}$ is the Frobenius norm. For a linear theory, only the terms that are of order $o(||\mathbf{D}||_{\mathbb{F}})$are kept. It should be noted that the third term $\mathbf{D}^T\mathbf{D}$ introduces a quadratic term of order $o^2(||\mathbf{D}||_{\mathbb{F}})$ and is negligible. Therefore **C** is expressed as

$$\mathbf{C} \;\; = \;\; \mathbf{F}^T\mathbf{F} \approx \mathbf{D} + \mathbf{D}^T + \mathbf{I} \tag{16}$$

3.4 Green-St. Venant strain tensor

Following, the Green-St.Venant tensor is calculated.

$$\mathbf{E} = \frac{1}{2}\Big(\mathbf{C} - \mathbf{I}\Big) = \frac{1}{2}\Big(\mathbf{D} + \mathbf{D}^T\Big) = \begin{bmatrix} 0 & 0 & \frac{1}{2}(\mathbf{D})_{31} \\ 0 & 0 & \frac{1}{2}(\mathbf{D})_{32} \\ \frac{1}{2}(\mathbf{D})_{31} & \frac{1}{2}(\mathbf{D})_{32} & (\mathbf{D})_{33} \end{bmatrix} \tag{17}$$

3.5 3D constitutive for an elastic material

The derivation of the following 3D constitutive relation for an elastic material can be in any textbook from continuum mechanics such as (Gurtin et al., 2009). The parameters E_y, $\mu = G_y$, λ, ν are elastic constants.

$$\mathbf{T}^{RR} = 2\mu\,\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\,\mathbf{I}, \quad \mu = \frac{E_y}{2(1+\nu)}, \quad \lambda = \frac{\nu\,E_y}{(1+\nu)(1-2\nu)} \tag{18}$$

3.6 First Piola Tensor - Elastic material

The expression for the first Piola stress tensor is obtaned in what follows. With the previous hypothesis, the deformation gradient tensor is stated as

$$\mathbf{F} = \mathbf{Q}\Big(\mathbf{D} + \mathbf{I}\Big) \approx \mathbf{Q} \tag{19}$$

Then if $\nu = 0$, the hypothesis is compatible with the fact that cross-sections behave as rigid bodies, $\lambda = 0$ and $\mu = E_y/2 = G_y$.

$$\mathbf{T}^R = \mathbf{F}\,\mathbf{T}^{RR} = \mathbf{Q}\,\mathbf{T}^{RR} = \mathbf{Q}\,E_y\,\mathbf{E} = \begin{bmatrix} 0 & 0 & G_y(\mathbf{D})_{31} \\ 0 & 0 & G_y(\mathbf{D})_{32} \\ G_y(\mathbf{D})_{31} & G_y(\mathbf{D})_{32} & E_y(\mathbf{D})_{33} \end{bmatrix} \tag{20}$$

Next, the traction vector **t** is obtained and integrated over the reference cross-section A^o, in order to find the expressions for the internal forces **n** and moments **m** acting at each point of the rod.

$$\mathbf{t}^R = \mathbf{T}^R \boldsymbol{\eta}^R = \mathbf{T}^R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{Q}(s^o) \begin{bmatrix} G_y \ (\mathbf{D})_{31} \\ G_y \ (\mathbf{D})_{32} \\ E_y \ (\mathbf{D})_{33} \end{bmatrix} \tag{21}$$

$$\mathbf{n}^R = \int_{A^o} \mathbf{t}^R \, dA^o = \mathbf{Q} \int_{A^o} \mathbf{t}^{RR} \, d\zeta_1 d\zeta_2 = \mathbf{Q} \, \mathbf{n}^{RR} \tag{22}$$

From (13), the vector $\mathbf{D}_3$ has the following expression

$$\mathbf{D}_3 = \left(\mathbf{v}^{RR} - \mathbf{e}_3 \right) + \left(\tilde{\mathbf{u}} \times \mathbf{z}^o \right) \tag{23}$$

For the case of a cross-section with double symmetry with respect to $\{\mathbf{e}_1, \mathbf{e}_2\}$, the area integral of the third term of (23) results in an even function evaluated in symmetric boundaries, and therefore vanishes.

$$\int_{A^o} \tilde{\mathbf{u}} \times \mathbf{z}^o \, d_{\zeta_1} d_{\zeta_2} = (\tilde{\mathbf{u}}_\times) \int_{A^o} \mathbf{z}^o \, d_{\zeta_1} d_{\zeta_2} = 0 \tag{24}$$

and the following expressions also hold

$$\int_{A^o} (\mathbf{v}^{RR} - \mathbf{e}_3) \ \zeta_1 \ d_{\zeta_1} d_{\zeta_2} = \int_{A^o} (\mathbf{v}^{R} - \mathbf{e}_3) \ \zeta_2 \ d_{\zeta_1} d_{\zeta_2} = 0 \tag{25}$$

The internal forces and moments acting on the cross-section are calculated hereunder.

$$\mathbf{n}^{RR} = \begin{bmatrix} \int_{A^o} G_y \ (\mathbf{D})_{31} d_{\zeta_1} d_{\zeta_2} \\ \int_{A^o} G_y \ (\mathbf{D})_{32} d_{\zeta_1} d_{\zeta_2} \\ \int_{A^o} E_y \ (\mathbf{D})_{33} d_{\zeta_1} d_{\zeta_2} \end{bmatrix} = \begin{bmatrix} G_y \ A \left((\mathbf{v}^{RR})_1 - (\mathbf{e}_3)_1 \right) \\ G_y \ A \left((\mathbf{v}^{RR})_2 - (\mathbf{e}_3)_2 \right) \\ E_y \ A \left((\mathbf{v}^{RR})_3 - (\mathbf{e}_3)_3 \right) \end{bmatrix} = \begin{bmatrix} G_y \ A \left((\mathbf{v}^{RR})_1 - 0 \right) \\ G_y \ A \left((\mathbf{v}^{RR})_2 - 0 \right) \\ E_y \ A \left((\mathbf{v}^{RR})_3 - 1 \right) \end{bmatrix} \tag{26}$$

$$\mathbf{m}^{RR} = \begin{bmatrix} \int_{A^o} E_y \ (\mathbf{D})_{33} \zeta_2 d_{\zeta_1} d_{\zeta_2} \\ -\int_{A^o} E_y \ (\mathbf{D})_{33} \zeta_1 d_{\zeta_1} d_{\zeta_2} \\ \int_{A^o} \left(G_y \ (\mathbf{D})_{32} \zeta_1 - G_y \ (\mathbf{D}_{31}) \zeta_2 \right) d_{\zeta_1} d_{\zeta_2} \end{bmatrix} = \begin{bmatrix} E_y \ J_{11} \ (\tilde{\mathbf{u}})_1 \\ E_y \ J_{22} \ (\tilde{\mathbf{u}})_2 \\ G_y \ J_0 \ (\tilde{\mathbf{u}})_3 \end{bmatrix} \tag{27}$$

Finally, $\mathbf{n}^{RR}$ and $\mathbf{m}^{RR}$ are the desired constitutive relations expressed in the material frame, as expressed in (26) and (27).

4 CONSTITUTIVE FOR A COSSERAT ROD WITH A LINEAR PIEZOELECTRIC MATERIAL

The 3D constitutive relations for some piezoelectric materials are presented in (Yang, 2005).

$$\begin{aligned} \mathbf{T}^{RR} &= \mathbb{C}^{\mathcal{E}} \mathbf{E} - e^T \boldsymbol{\mathcal{E}}^{RR} \\ \boldsymbol{\mathcal{D}}^{RR} &= e\mathbf{E} + \epsilon^E \boldsymbol{\mathcal{E}}^{RR} \end{aligned} \tag{28}$$

Regular notation	$(\cdot)_{ij}$ or $(\cdot)_{kl}$:	$(\cdot)_{11}$	$(\cdot)_{22}$	$(\cdot)_{33}$	$(\cdot)_{23}$ or $(\cdot)_{32}$	$(\cdot)_{13}$ or $(\cdot)_{31}$	$(\cdot)_{12}$ or $(\cdot)_{21}$
Voigt notation	$(\cdot)_p$ or $(\cdot)_q$:	$(\cdot)_1$	$(\cdot)_2$	$(\cdot)_3$	$(\cdot)_4$	$(\cdot)_5$	$(\cdot)_6$

Table 1: Index notation convention to represent $\mathbb{C}_{ijkl}$, e_{ikl} into the reduced Voigt notation form $\mathbb{C}_{pq}$, e_{iq}.

In general, the form of the constitutive law can be expressed as in (28), where $\mathbb{C}^E$ is the elastic moduli tensor, e is the piezoelectric constants tensor, and ϵ^S are dielectric constants. The superscript $\mathcal{E}$ in $\mathbb{C}^{\mathcal{E}}$ indicates that the independent electric constitutive variable is the electric field. The superscript e^E indicates that the mechanical constitutive variable is the strain (Green-St. Venant) tensor $\mathbf{E}$.

Next, the previous equations are expressed in matrix form. For this task, the following compact notation $\mathbb{C}_{ijkl} \to \mathbb{C}_{pq}$ and $e_{ikl} \to e_{ip}$ has been employed, where the indices ij or kl are transformed into p or q indices following the Voigt notation indicated in Table 1.

For the particular case of a ceramic poled along the x-axis (or 1-axis in our current notation), the constitutive relation is expressed in matrix below.

$$\begin{bmatrix} (\mathbf{T}^{RR})_{11} \\ (\mathbf{T}^{RR})_{22} \\ (\mathbf{T}^{RR})_{33} \\ (\mathbf{T}^{RR})_{23} \\ (\mathbf{T}^{RR})_{13} \\ (\mathbf{T}^{RR})_{12} \\ (\boldsymbol{\mathcal{D}}^{RR})_1 \\ (\boldsymbol{\mathcal{D}}^{RR})_2 \\ (\boldsymbol{\mathcal{D}}^{RR})_3 \end{bmatrix} = \left[\begin{array}{cccccc|ccc} \mathbb{C}^{\mathcal{E}}_{33} & \mathbb{C}^{\mathcal{E}}_{13} & \mathbb{C}^{\mathcal{E}}_{13} & 0 & 0 & 0 & -e_{33} & 0 & 0 \\ \mathbb{C}^{\mathcal{E}}_{13} & \mathbb{C}^{\mathcal{E}}_{11} & \mathbb{C}^{\mathcal{E}}_{12} & 0 & 0 & 0 & -e_{13} & 0 & 0 \\ \mathbb{C}^{\mathcal{E}}_{13} & \mathbb{C}^{\mathcal{E}}_{12} & \mathbb{C}^{\mathcal{E}}_{11} & 0 & 0 & 0 & -e_{13} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C}^{\mathcal{E}}_{66} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{C}^{\mathcal{E}}_{44} & 0 & 0 & 0 & -e_{15} \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}^{\mathcal{E}}_{44} & 0 & -e_{15} & 0 \\ \hline e_{33} & e_{31} & e_{31} & 0 & 0 & 0 & \epsilon^E_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{15} & 0 & \epsilon^E_{11} & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & 0 & 0 & \epsilon^E_{11} \end{array}\right] \begin{bmatrix} (\mathbf{E})_{11} \\ (\mathbf{E})_{22} \\ (\mathbf{E})_{33} \\ (\mathbf{E})_{23} \\ (\mathbf{E})_{13} \\ (\mathbf{E})_{12} \\ (\boldsymbol{\mathcal{E}}^{RR})_1 \\ (\boldsymbol{\mathcal{E}}^{RR})_2 \\ (\boldsymbol{\mathcal{E}}^{RR})_3 \end{bmatrix} \quad (29)$$

The beam will be considered built in multiple layers as shown in Fig. 3. The position of the face i-th interface between layers is denoted by κ_i.

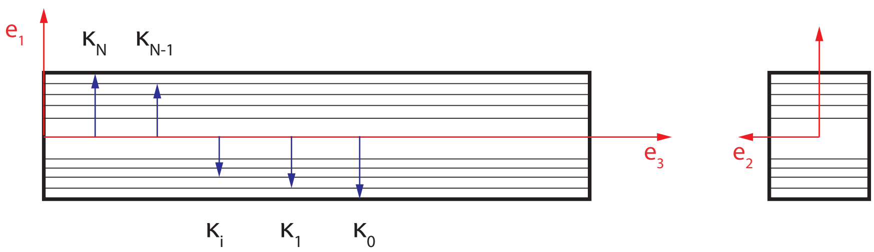

Figure 3: Sketch. Piezoelectric layers in a Cosserat rod.

Now, going back to tensor notation, the second Piola stress $\mathbf{T}^{RR}$ and the material pullback of the traction vector $\mathbf{t}^{RR}$ can be written as follows

$$\mathbf{T}^{RR} = \begin{bmatrix} \mathbb{C}^{\mathcal{E}}_{13}(\mathbf{E})_{33} - e_{33}\boldsymbol{\mathcal{E}}^{RR}_1 & 0 & \mathbb{C}^{\mathcal{E}}_{44}(\mathbf{E})_{13} \\ 0 & \mathbb{C}^{\mathcal{E}}_{12}(\mathbf{E})_{33} - e_{31}\boldsymbol{\mathcal{E}}^{RR}_1 & \mathbb{C}^{\mathcal{E}}_{66}(\mathbf{E})_{23} \\ \mathbb{C}^{\mathcal{E}}_{44}(\mathbf{E})_{13} & \mathbb{C}^{\mathcal{E}}_{66}(\mathbf{E})_{23} & \mathbb{C}^{\mathcal{E}}_{11}(\mathbf{E})_{33} - e_{31}\boldsymbol{\mathcal{E}}^{RR}_1 \end{bmatrix} \quad (30)$$

$$\mathbf{t}^{RR} = \mathbf{T}^{RR}\,\boldsymbol{\eta}^{R} \begin{bmatrix} \mathbb{C}^{\mathcal{E}}_{44}(\mathbf{D})_{31} \\ \mathbb{C}^{\mathcal{E}}_{66}(\mathbf{D})_{32} \\ \mathbb{C}^{\mathcal{E}}_{11}(\mathbf{D})_{33} - e_{31}(\boldsymbol{\mathcal{E}}^{RR})_1 \end{bmatrix} = \begin{bmatrix} \mathbb{C}^{E}_{44}(\mathbf{D})_{31} \\ \mathbb{C}^{E}_{66}(\mathbf{D})_{32} \\ \mathbb{C}^{E}_{11}(\mathbf{D})_{33} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -e_{31}(\boldsymbol{\mathcal{E}}^{RR})_1 \end{bmatrix} \tag{31}$$

Integrating the previous expression for the cross-section, the constitutive relations for the internal forces in a Cosserat medium are obtained.

$$\mathbf{n}^{R} = \int_{A^o} \mathbf{t}^{R}\, dA^o = \mathbf{Q} \int_{A^o} \mathbf{t}^{RR}\, d\zeta_1 d\zeta_2 = \mathbf{Q}\,\mathbf{n}^{RR} \tag{32}$$

The traction $\mathbf{t}^{RR}$ is composed of a mechanical term and an electrical term, as shown in (31). In particular, the first integral has already been solved for the case of a pure elastic rod in (26), so only the second term remains to be analysed. Remembering that the electric field can be derived from a potencial function, $\boldsymbol{\mathcal{E}}^{RR}_1 = -\dfrac{\partial\phi}{d\zeta_1}$, and that the electric field is constant

$$\int_{A^o} -e_{31}\boldsymbol{\mathcal{E}}^{RR}_1 d\zeta_1\zeta_2 \;=\; \int_{A^o} e_{31}\frac{\partial\phi}{d\zeta_1} d\zeta_1\zeta_2 = \sum_i e_k b\, V_k \tag{33}$$

$$\mathbf{n}^{RR} = \begin{bmatrix} \mathbb{C}^{E}_{44}\, A \left((\mathbf{v}^{RR})_1 - 0 \right) \\ \mathbb{C}^{E}_{66}\, A \left((\mathbf{v}^{RR})_2 - 0 \right) \\ \mathbb{C}^{E}_{11}\, A \left((\mathbf{v}^{RR})_3 - 1 \right) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sum_i e_k b\, V_k \end{bmatrix} \tag{34}$$

Next, the moments produced by the previous stresses are calculated.

$$\mathbf{m}^{RR} = \begin{bmatrix} \int_{A^o} \left(\mathbb{C}^{\mathcal{E}}_{11}(\mathbf{D})_{33} - e_{31}(\boldsymbol{\mathcal{E}}^{RR})_1 \right) \zeta_2\, d\zeta_1 d\zeta_2 \\ \int_{A^o} -\left(\mathbb{C}^{\mathcal{E}}_{11}(\mathbf{D})_{33} - e_{31}(\boldsymbol{\mathcal{E}}^{RR})_1 \right) \zeta_1\, d\zeta_1 d\zeta_2 \\ \int_{A^o} \left(-\left(\mathbb{C}^{\mathcal{E}}_{66}(\mathbf{D})_{31} \right) \zeta_2 + \left(\mathbb{C}^{\mathcal{E}}_{44}(\mathbf{D})_{32} \right) \zeta_1 \right) d\zeta_1 d\zeta_2 \end{bmatrix} \tag{35}$$

Once again, the previous expression can be divided into a mechanical and an electrical effect. The mechanical term has already been solved in (27). Then, considering that $\boldsymbol{\mathcal{E}}^{RR}_1$ is constant within each layer, and that each piezoelectric patch is located at $\kappa_{i-1} \leq \zeta_1 \leq \kappa_i$ and $-b_i/2 \leq \zeta_2 \leq b_i/2$, with $h_i = \kappa_i - \kappa_{i-1}$, $i = \{0, 1, ..., N\}$, and $l_i \leq S^0 \leq l_j$, where l_i and l_j define the location of the piezoelectric patch in relation to the reference arc-length.

$$-\sum_i \int_{\kappa_{i-1}}^{\kappa_i} \int_{-b_i/2}^{b_i/2} e_{31}(\boldsymbol{\mathcal{E}}^{RR})_1 \zeta_1 d\zeta_2 d\zeta_1 \;=\; \sum_i b_i e_i \frac{\partial\phi}{\partial\zeta_1} \frac{\zeta_2^2}{2}\bigg|_{\kappa_{i-1}}^{\kappa_i} = \sum_i e_i b_i \frac{\kappa_{i-1} + \kappa_i}{2} V_i \tag{36}$$

$$\mathbf{m}^{RR} = \begin{bmatrix} E_y\; J_{11}\; (\tilde{\mathbf{u}})_1 \\ E_y\; J_{22}\; (\tilde{\mathbf{u}})_2 \\ G_y\; J_0\; (\tilde{\mathbf{u}})_3 \end{bmatrix} - \begin{bmatrix} 0 \\ \sum_i e_i b_i \dfrac{x_{i-1} + x_i}{2} V_i \\ 0 \end{bmatrix} \tag{37}$$

Equations (34) and (37) provide the coupled electro-mechanical constitutive equations for the linear and momentum balances in Cosserat rods.

Finally, to complete the derivation, the required constitutive relations for the electrical circuit are sought. The hypothesis that $e_{15} = 0$ is used, which is consistent with the material PIC151 (PICeramic, 2018) employed in (Ducarne, 2009).

$$\boldsymbol{\mathcal{D}}^{RR} = \begin{bmatrix} e_{31}(\mathbf{E})_{33} + \epsilon_{33}^{E}(\boldsymbol{\mathcal{E}}^{RR})_1 \\ 0 \\ e_{15}(\mathbf{E})_{13} \end{bmatrix} = \begin{bmatrix} e_{31}(\mathbf{E})_{33} + \epsilon_{33}^{E}(\boldsymbol{\mathcal{E}}^{RR})_1 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

The charge Q_k of the k-th piezoelectric patch is, by definition, the amount of free electrical charges within one of the electrodes. For every patch, the superior electrode is chosen to perform the calculation.

Applying the Gauss theorem in terms of charge displacement, the following equation holds

$$Q_k = \oiint \boldsymbol{\mathcal{D}}^{RR} \cdot \boldsymbol{\eta}^R dA^o = \iiint \rho_v^R d\Omega^R \tag{39}$$

The vector $\boldsymbol{\mathcal{D}}^{RR}$ vanishes inside the electrode. In order to calculate the previous integral, a Gaussian pillbox in the vicinity of the surface of the electrode at $\kappa^- < \kappa_k < \kappa^+$ is considered. In the present analysis, it is considered that the piezoelectric patches are placed in such way that electrodes are not shared among piezoelectric layers, nor in contact with an insulator where a non-negligible charge displacement exists. With such considerations, the previous integral vanishes at every face of the pillbox but the lower side, whose exterior normal is $\boldsymbol{\eta}^R = (-1, 0, 0)'$.

$$\begin{aligned} Q_k &= \iint -(\boldsymbol{\mathcal{D}}^{RR})_1 d\zeta_2 ds^o \\ &= -\iint \left(e_{31}(\mathbf{E})_{33} + \epsilon_{33}^{E}(\boldsymbol{\mathcal{E}}^{RR})_1 \right) d\zeta_2 ds^o = \mathbb{I}_1 + \mathbb{I}_2 \end{aligned} \tag{40}$$

In what follows, the following reduced nomenclature will be used for the material properties of the k-th layer: $e_k = e_{31k}$, $\epsilon_k = \epsilon_{33}{}_k^E$, $\mathfrak{C}_k = \dfrac{\epsilon_k l_k b}{h_k}$, $\Xi_k = b e_k$.

$$\mathbb{I}_2 = -\int_{l_i}^{l_j} \int_{-b_i/2}^{b_i/2} \epsilon_{33}^{E}(\boldsymbol{\mathcal{E}}^{RR})_1 d\zeta_2 ds^o = \int_{l_i}^{l_j} \epsilon_k b \frac{d\phi}{d\zeta_1} ds^o \tag{41}$$

$$\begin{aligned} \mathbb{I}_1 &= -\int_{l_i}^{l_j} \int_{-b_i/2}^{b_i/2} e_{31}(\mathbf{E})_{33} d\zeta_2 ds^o \\ &= -\int_{l_i}^{l_j} \int_{-b_i/2}^{b_i/2} e_{31}\left(\mathbf{v}^R - \mathbf{e}_3\right)_3 d\zeta_2 ds^o - \int_{l_i}^{l_j} \int_{-b_i/2}^{b_i/2} e_{31}\left(\tilde{\mathbf{u}} \times \mathbf{z}^o\right)_3 d\zeta_2 ds^o \\ &= \mathbb{I}_{1A} + \mathbb{I}_{1B} \end{aligned} \tag{42}$$

$$\begin{aligned} \mathbb{I}_{1A} &= -\int_{l_i}^{l_j} \int_{-b_i/2}^{b_i/2} b e_k (\mathbf{v}^R - \mathbf{e}_3)_3 ds^o \\ &= -\int_{l_i}^{l_j} \Xi_k (\mathbf{v}^R - \mathbf{e}_3)_3 ds^o \end{aligned} \tag{43}$$

$$\begin{aligned} \mathbb{I}_{1B} &= -\int_{l_i}^{l_j} \int_{-b_i/2}^{b_i/2} e_{31}\left(\tilde{\mathbf{u}} \times \mathbf{z}^o\right)_3 d\zeta_2 ds^o \\ &= \int_{l_i}^{l_j} b e_k (\tilde{\mathbf{u}}_2) \zeta_1 ds^o \end{aligned} \tag{44}$$

$$
\begin{aligned}
Q_k &= \mathbb{I}_{1A} + \mathbb{I}_{1B} + \mathbb{I}_2 \\
&= -\int_{l_i}^{l_j} \Xi_k (\mathbf{v}^R - \mathbf{e}_3)_3 \;\; ds^o + \int_{l_i}^{l_j} \Xi_k \zeta_1 (\tilde{\mathbf{u}})_2 \; + \epsilon_k l_k b \frac{d\phi}{d\zeta_1}
\end{aligned} \tag{45}
$$

Integrating over the height of the patch, an expression for the potential difference generated by the presence of the piezoelectric is obtained. This is the constitutive relation (coupling term) that will be introduced in an electrical circuit that is connected to the patch.

$$
V_k = \frac{1}{\mathfrak{C}_k} Q_k + \frac{1}{\mathfrak{C}_k} \int_{l_i}^{l_j} \Xi_k (\mathbf{v}^{RR} - \mathbf{e}_3)_3 ds^o - \frac{1}{\mathfrak{C}_k} \int_{l_i}^{l_j} \Xi_k \frac{x_i + x_{i-1}}{2} (\tilde{\mathbf{u}})_2 ds^o \tag{46}
$$

5 CONCLUSIONS

The constitutive relations for both an elastic medium and a piezoelectric layer have been derived, which allows modelling a rod with both elastic materials and piezoelectric patches. The expressions for the constitutive relations the elastic layers are given by (26) and (27), and for the piezoelectric patches (or layers) they follow (34), (37) and (46). Furthermore, it is observed that the mathematical expressions for the constitutive relations on a piezoelectric patch has the same structure as that of an elastic material, if the coupled electrical terms are neglected.

REFERENCES

Cao D.Q. and Tucker R.W. Nonlinear dynamics of elastic rods using the Cosserat theory: Modelling and simulation. *International Journal of Solids and Structures*, 45(2):460–477, 2008. ISSN 00207683. doi:10.1016/j.ijsolstr.2007.08.016.

Ducarne J. *Modélisation et optimisation de dispositifs non-linéaires d'amortissement de structures par systèmes piézoélectriques commutés*. Ph.D. thesis, Conservatoire national des arts et metier, 2009.

Erturk A. and Inman D.J.I. *Piezoelectric Energy Harvesting*. John Wiley & Sons, Inc., Chichester, 2011. ISBN 9780470682548.

Géradin M. and Cardona A. *Flexible Multibody Dynamics. A finite element approach.* Chichester, 2001. ISBN 0471489905.

Goicoechea H.E., Buezas F.S., and Rosales M.B. A non-linear Cosserat rod model for drill-string dynamics in arbitrary borehole geometries with contact and friction. *International Journal of Mechanical Sciences*, 157-158:98–110, 2019. ISSN 00207403. doi:10.1016/j.ijmecsci.2019.04.023.

Gurtin M.E., Fried E., and Anand L. *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, Cambridge, 2009. ISBN 9781107617063.

Linn J., Lang H., and Tuganov A. Geometrically exact cosserat rods with Kelvin-Voigt type viscous damping. *Mechanical Sciences*, 4(1):79–96, 2013. ISSN 21919151.

PICeramic. Piezoelectric Ceramic Products Brochure by PI ceramic. 2018.

Rao S.S. and Sunar M. Piezoelectricity and Its Use in Disturbance Sensing and Control of Flexible Structures: A Survey. *Applied Mechanics Reviews*, 47(4):113, 2009. ISSN 00036900.

Yang J. *An Introduction to the Theory of Piezoelectricity*. Springer, 2005. ISBN 978-0-387-23546-2.