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AN INFINITE DIMENSIONAL TECHNIQUE TO ESTIMATE OIL-WATER SATURATION FUNCTIONS

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Abstract. We introduce an algorithm to solve an inverse problem for a non-linear system of partial differential equations, which can be used to estimate oil water displacement functions. The direct model is non-linear because the sought for parameter is a function of the solution of the system of equations. Traditionally, the estimation of functions requires the election of a fitting parametric model and thus the optimum curve depends on that election. We develop an algorithm that does not require a parametric model and thus provides a more objective fit. The estimation procedure is carried out linearizing the solution of the direct model with respect to the parameter and then computing the least squares solution in functional spaces. We present the partial differential equations that are used to compute the Fréchet derivative. The resulting method has shown convergence in numerical tests, and because of its general theoretical formulation has the potential to be extended to solve more complex problems. The main contribution of this work is the formulation and application of the algorithm described above to estimate non-linear parameters in functional spaces. This algorithm obtains the sought-after parameters without the imposition of a priori parametric models. Though the use of such models is currently the common practice among field engineers, different models yield different results and there is no objective criterion to choose among them.

1 INTRODUCTION

The purpose of this paper is to estimate relative permeabilities curves from measurements taken during a displacement test of oil by water at the laboratory. These curves appear as coefficients of the equations that rule the two phase flow through a porous media. Fluid saturations and pressures are the solution of the differential problem. As the oil-water relative permeability curves are functions of saturations, the direct model is a non-linear partial differential system.

The traditional approach to estimate a function (e.g. relative permeability) requires a parametric model depending on constant values. These values are determined minimizing in finite dimensional spaces (Chardaire-Riviere et al., 1992; Savioli and Bidner, 1994, 1995). This approach has the drawback of imposing a subjective parametric model. Some authors have proposed alternatives to deal with the estimation problem without imposing a parametric model, by using discretized direct models (Kruger et al., 2003; Valestrand et al., 2003).

This work proposes a new alternative to the traditional approach, which is novel in the field of non-linear equations. Up to now, it has been successfully applied to problems where the direct model is based on a linear partial differential equation (Fernández-Berdaguer, 1998; Fernández-Berdaguer et al., 1995, 1996; Tarantola, 1987).

2 THE MATHEMATICAL DIRECT MODEL

Accurate numerical simulation is a crucial task to predict the flow pattern on a reservoir. Essential to the simulation are the values of the parameters appearing in the model. Mathematical flow models consist on a strongly coupled system of non-linear equations.

In this section we present a model for two-phase, incompresible immiscible flow (H. J. Schroll and A. Tveito, 1997). We will ignore the capillary pressure and assume that pressures in the oil and water phases are equal.

Since we are assuming that the flow and the rock are incompressible the densities $\rho_i,\ i = o, w$ and the porosity Φ are independent of the pressure. We denote by K the absolute permeability, k_{ri} denotes the the relative permeability of phase i and μ_i the viscosity of phase i. The absolute permeability, the porosity and the viscosities are considered constant parameters, wehereas the relative permeabilities for oil and water are functions of the oil saturation only. We define λ_i as

$$\lambda_i = KA\frac{k_{ri}}{\mu_i},$$

where A denotes the cross-sectional area for the flow.

The total mobility is $\lambda_T = \lambda_w + \lambda_o$, and the fractional flow function of phase o is $f = \dfrac{\lambda_o}{\lambda_T}$, thus the equations that rule the two phase flow are

$$\begin{aligned}
&\Phi\frac{\partial S(x,t)}{\partial t} + V_T(x,t)\frac{\partial f(S(x,t))}{\partial x} = 0,\\
&\frac{\partial}{\partial x}(\lambda_T\big(S(x,t)\big)\frac{\partial P(x,t)}{\partial x}) + AQ_T = 0,\\
&AV_T(x,t) + \lambda_T\big(S(x,t)\big)\frac{\partial P(x,t)}{\partial x} = 0, \quad x \in (0,L),\, t \in (0,T].
\end{aligned} \tag{1}$$

Here, Q_T is the source term, total volumetric flow rate per unit volume and V_T is the total Darcy velocity. Notice that $V_T = V_T(\lambda_T, P)$.

The first equation is the saturation equation, the second the pressure equation.

In the case of water injection the initial conditions is

$$S(x,0) = \begin{cases} S_{or}, & x = 0 \\ 1 - S_{wc}, & 0 < x \leq 1 \end{cases} \tag{2}$$

and boundary conditions for known injection flow rate are

$$Q_{wi}(t) = KA \frac{k_{rw}\big(S(0,t)\big)}{\mu_w} \frac{\partial P}{\partial x}(0,t), \tag{3}$$

$$P(1,t) = p(t). \tag{4}$$

2.1 A simpler case

In this paper we will handle the case of smooth solutions of the problem. In order to do that we must assume that the initial condition for the saturation is a continuous approximation $g(x)$ to the physically correct $S(x,0)$ (2).

Also we assume constant inyection water flow rate, thus the velocity V_T is constant. Source terms are nil in the equations because they are considered in the inlet and outlet boundary conditions.

Then the direct model that we consider is the initial-boundary value problem (5)

$$\begin{aligned} \Phi \frac{\partial S(x,t)}{\partial t} + H(S(x,t)) \frac{\partial S(x,t)}{\partial x} &= 0, \\ \frac{\partial}{\partial x}\big(\lambda_T(x,t) \frac{\partial P(x,t)}{\partial x}\big) &= 0,\ x \in (0,L),\ t \in (0,T], \end{aligned} \tag{5}$$

where $H(S) = \dfrac{Q_{wi}}{AL} f'(S)$.

The initial condition is

$$S(x,0) = g(x), \quad x \in [0,L], \tag{6}$$

and inlet boundary condition is constant injection water rate,

$$Q_{wi} = \lambda_T \frac{\partial P}{\partial x}\Big|_{x=0}. \tag{7}$$

At outlet ($x = 1$), the boundary condition is imposed by the atmospheric pressure,

$$P(1,t) = p_a,\ t \in (0,T]. \tag{8}$$

3 THE ESTIMATION PROBLEM

The measurements, which are denoted by $S^{obs}(t)$ and $P^{obs}(t)$, are the values of the saturation and pressure at a recording point x^{rec}, for times in the interval $[0,T]$. The problem is to find (λ_T^*, H^*) such that $(S,P)(\lambda_T^*, H^*)(x^{rec}, t)$ matches the values of $(S^{obs}, P^{obs})(t)$, for $t \in [0,T]$. Specifically: find (λ_T^*, H^*) such that

$$S(\lambda_T^*, H^*)(x^{rec}, t) = S^{obs}(t), \tag{9}$$

$$P(\lambda_T^*, H^*)(x^{rec}, t) = P^{obs}(t). \tag{10}$$

The first equation is non-linear in the arguments whereas the second one is linear on $\dfrac{1}{\lambda_T}$ because it is equivalently written as:

$$p_a + \int_{x^{rec}}^{L} \frac{Q_{wi}}{\lambda_T(S(\eta,t))}\, d\eta = P^{obs}(t). \tag{11}$$

We will linearize the saturation equation in order to obtain a linear system. We denote by S'_{λ_T}, S'_H the derivatives of the function S with respect to the parameters λ_T, H respectively.

Let $\lambda = \tilde{\lambda_T} + \delta\lambda$ and $H = \tilde{H} + \delta H$, the linearized function S about $(\tilde{\lambda_T}, \tilde{H})$ is calculated as

$$S(\lambda_T, H) = S(\tilde{\lambda_T}, \tilde{H}) + S'_{\lambda_T}(\tilde{\lambda_T}, \tilde{H})\,\delta\lambda_T + S'_H(\tilde{\lambda_T}, \tilde{H})\,\delta H. \tag{12}$$

Now, the approximate problem to solve is the set of equations formed by the linearized saturation equation

$$\Big(S(\tilde{\lambda_T}, \tilde{H}) + S'_{\lambda_T}(\tilde{\lambda_T}, \tilde{H})\,\delta\lambda_T + S'_H(\tilde{\lambda_T}, \tilde{H})\,\delta H\Big)(x^{rec}, t) = S^{obs}(t) \tag{13}$$

and the pressure equation (11).

3.1 Sensitivity equations

In this section we describe how to compute the terms $S'_{\lambda_T}(\tilde{\lambda_T}, \tilde{H})\,\delta\lambda_T$ and $S'_H(\tilde{\lambda_T}, \tilde{H})\,\delta H$ in (13). That is, how to compute the Fréchet derivatives of S with respect to H and λ_T.

We denote by Λ the pair of functions (λ_T, H).

For each (λ_T, H) there is a solution of (5)-(8). To derive the sensitivity equation for S'_{λ_T} and S'_H we consider the function

$$F\Big(\Lambda, S(\Lambda), P(\Lambda)\Big) = \Phi \frac{\partial S(\Lambda)}{\partial t} + H(S(\Lambda)) \frac{\partial S(\Lambda)}{\partial x}. \tag{14}$$

The equation

$$F\big(\Lambda, S(\Lambda), P(\Lambda)\big) = 0 \tag{15}$$

defines implicitly S, P as functions of Λ. Then we compute the Fréchet derivatives of F with respect to (λ_T, H) from (15) to obtain a system of differential equations for S'_{λ_T}, S'_H:

$$\begin{aligned} \frac{\partial F}{\partial \lambda_T} &= \Phi \frac{\partial S'_{\lambda_T}}{\partial t} + H \frac{\partial S'_{\lambda_T}}{\partial x} = 0, \\ \frac{\partial F}{\partial H} &= \Phi \frac{\partial S'_H}{\partial t} + H \frac{\partial S'_H}{\partial x} + \frac{\partial S}{\partial x} = 0. \end{aligned} \tag{16}$$

Since in the direct model the initial condition is independent of Λ the initial condition for (16) is zero. Thus the solution of the first equation above $S'_{\lambda_T}(x,t),\ x \in (0,L),\ t \in (0,T]$ is zero and as a consequence we only have to solve the sensitivity equation for S'_H:

$$\Phi \frac{\partial S'_H}{\partial t} + H(S) \frac{\partial S'_H}{\partial x} = -\frac{\partial S}{\partial x}\,. \tag{17}$$

The above equation makes clear that there is no dependency of S'_H on λ_T. Thus in the notation we will supress that dependency and we will use the notation $S'_H(H)$.

The Fréchet derivative aplied to δH, $\Big(S'_H(\tilde{H})\,\delta H\Big)(x^{rec}, t)$, is computed as

$$\Big(S'_H(\tilde{H})\,\delta H\Big)(x^{rec}, t) = S'_H(x,t)\delta H(S(x,t)). \tag{18}$$

4 THE ESTIMATION ALGORITHM

From the previous section we have the linear approximation to (9)

$$\Big(S'_H(\tilde{H})\,\delta H\Big)(x^{rec},t) = S^{obs}(t) - S(\tilde{H})(x^{rec},t). \tag{19}$$

The algorithm to estimate H and $\dfrac{1}{\lambda_T}$ consists of two steps:

- First, we estimate H^* iteratively as
  1. Give an approximation $\widetilde{H}$ of H^*,
  2. Calculate an increment δH using the least squares method, that is, we solve the following normal equation
  $$\big(S'_H(\tilde{H})\big)^*\big(S'_H(\tilde{H})\,\delta H\big)(x^{rec},t) = \big(S'_H(\tilde{H})\big)^*\big(S^{obs}(t) - S(\tilde{H})(x^{rec},t)\big). \tag{20}$$
  3. Update H as $H_{new} = \widetilde{H} + \delta H$.
- Second, once H^* has been estimated, we obtain an approximation to λ_T^* solving
$$p_a + \int_{x^{rec}}^{L} \frac{Q_{wi}}{\lambda_T(S(\eta,t))}\,d\eta = P^{obs}(t). \tag{21}$$

The above equation is a Volterra type equation as we show in subsection (4.2). Following we describe the discretization to carry out the numerical implementation of the continuous algorithm.

4.1 Estimation of the parameter function H

First we approximate the function H by finite elements to make it amenable to calculations. We seek functions $H(z)$ of the form

$$H(z) = \sum_{j=1}^{M} \gamma_j \psi_j(z). \tag{22}$$

The finite elements $\psi_j(z)$ that we use in (22)are defined as follows. We make a partition $\{z_j\},\ j = 1,\dots,M$; of the domain of H, such that $z_1 < z_2 < \cdots < z_M$. Now,

$$\psi_j(z) = \begin{cases} \dfrac{z - z_{j-1}}{z_j - z_{j-1}} & z_{j-1} \le z \le z_j \\ \dfrac{z_{j+1} - z}{z_{j+1} - z_j} & z_j \le z \le z_{j+1} \\ 0 & \text{otherwise}, \qquad j = 1,\dots,M. \end{cases} \tag{23}$$

4.1.1 Discretization of the normal equation

We give an initial guess of $H(z)$: H^0.

Next we specify the procedure for each iteration $k \geq 0$.

First we calculate $S(H^k)(x^{rec}, t)$. Since H^k is a piecewise linear function, the solution of (5)-(8) can be calculated by solving the following (non-linear in most cases) equation on $S(H^k)$:

$$S(H^k)(x^{rec}, t) = g\Big(x^{rec} - \frac{\gamma_j - \gamma_{j-1}}{z_j^{k-1} - z_{j-1}^{k-1}}\, S(H^k)(x^{rec}, t)\, t - \frac{\gamma_{j-1} z_j^{k-1} - \gamma_j z_{j-1}^{k-1}}{z_j^{k-1} - z_{j-1}^{k-1}}\, t\Big), \tag{24}$$

where j is chosen to satisfy

$$z_{j-1}^{k-1} \leq S(x^{rec}, t) \leq z_j^{k-1}. \tag{25}$$

Now, with $S(H^k)(x^{rec}, t)$ computed above, we define a partition of the domain of H^k by points $\{z_j^k\}$ as follows:

$$z_j^k = S(H^k)(x^{rec}, t_{M-j+1}), \quad j = 1, \ldots, M. \tag{26}$$

We notice that in the algorithm the nodes z_j and, as a consequence, the functions ψ_j of Eq.(23) change at each iteration k, to remark it we denote the coefficients and the functions in (22) by γ_j^k and ψ_j^k respectively.

Next, we express H^k and the unknown increment δH^k using the discretization (22). Explicitly, the increment δH^k is

$$\delta H^k(z) = \sum_{j=1}^{M} \delta\gamma_j^k \psi_j^k(z). \tag{27}$$

To determine $\delta\gamma_j^k,\ j = 1, \ldots, M$; we solve (20) for $S(\tilde{H}) = S(H^k)$ and $\delta H = \delta H^k$.

The computation of the discrete version of (20) requires tedious calculations, which are detailed in the Appendix and lead to the simple formula

$$\delta\gamma_j^k = -\frac{S(H^k)(x^{rec}, t_{M-j+1}) - S^{obs}(t_{M-j+1})}{S'_H(x^{rec}, t_{M-j+1})}. \tag{28}$$

By using (28) in (27) we compute δH^k and update H^k,

$$H^{k+1} = H^k + \delta H^k. \tag{29}$$

The stopping criterion is that the residual

$$J(H^k) = \Big(\int_0^T \big(S(H^k)(x^{rec}, \cdot) - S^{obs}\big)^2\, dt\Big)^{1/2} \tag{30}$$

must be small.

4.1.2 Outline of the algorithm

1. Read $(t_j, S_j^{obs}),\ j = 1, \ldots, M$ and the stopping criterion (TOL).

2. Give an initial guess H^0, set $k = 0$.

3. Solve the initial value problem (5)-(8) with $H = H^k$. Compute $S(H^k(x^{rec}, t_j))$.
4. Compute the residual $J(H^k)$, (Eq. (30)).
5. If $J(H^k) < \mathrm{TOL}$ then H^k is the solution. STOP
6. If not,

   for $j = 1, \ldots, M$

   - Define $z_j^k = S(H^k)(x^{rec}, t_{M-j+1})$
   - Compute $\gamma_j^k = H^k(z_j^k)$.
   - Compute $S'_H(x,t)$, solving the initial value problem (17) with $H = H^k$.
   - Compute $\delta\gamma_j^k$ applying (28).
   - Define $\gamma_j^{k+1} = \gamma_j^k + \delta\gamma_j^k$.

   end for

   - Update H: $H^{k+1} = \sum_{j=1}^{M} \gamma_j^{k+1} \psi_j^k(z)$.
   - $k = k + 1$, go to 3.

4.2 Estimation of the parameter function $\frac{1}{\lambda_T}$

Now, once H^{opt} has been estimated, we seek for $\frac{1}{\lambda_T^*}$ that satisfies

$$p_a + \int_{x^{rec}}^{L} \frac{Q_{wi}}{\lambda_T^*(S(H^{opt})(\eta, t))}\, d\eta = P^{obs}(t). \tag{31}$$

For $H = H^{opt}$ we apply the following change of variables:

$$z = S(H)(\eta, t). \tag{32}$$

Therefore

$$dz = S(H)_x(\eta, t)\, d\eta = \frac{g'\big(g^{-1}(z)\big)}{1 + g'\big(g^{-1}(z)\big) H'(z) t}\, d\eta. \tag{33}$$

Then (31) can be written as

$$P^{obs}(t) = P(x^{rec}, t) = q_a + \int_{S(x^{rec}, t)}^{S(L,t)} K(z,t) \big(\lambda_T^*(z)\big)^{-1}\, dz \tag{34}$$

where

$$K(z,t) = \frac{1 + g'\big(g^{-1}(z)\big) H'(z) t}{g'\big(g^{-1}(z)\big)} \tag{35}$$

4.2.1 Numerical computation of the parameter function $\frac{1}{\lambda_T}$

The discretization of function $\frac{1}{\lambda_T}$ is analogous to that of the function H,

$$\frac{1}{\lambda_T} = \sum_{j=1}^{M} \xi_j \psi_j(z), \tag{36}$$

where $\psi_j(z)$ are defined in (23).

Replacing (36) in

$$P^{obs}(t_j) = P(x^{rec}, t_j) = q_a + \int_{S^{obs}(t_j)}^{S(L,t_j)} K(z,t)\big(\lambda_T^*(z)\big)^{-1}\, dz, \tag{37}$$

we obtain a $M \times M$ upper triangular linear system in the unknowns ξ_j, $j = 1, \ldots, M$.

5 NUMERICAL EXPERIMENTS

The algorithm described in the previous section was tested with several examples. We estimate the derivative of the fractional flow curve: $H(z)$ and the inverse of the total mobility $1/\lambda_T$.

Two examples are shown. In both the fractional flow curve and the total mobility have a typical shape because we choose the well known potential model for oil and water relative permeabilities. In the first case, $k_{ro}(S) = S^2$ and $k_{rw}(S) = 0.2\,(1-S)^2$. In the second one $k_{ro}(S) = S^2$ and $k_{rw}(S) = 0.2\,(1-S)$. Besides the values of oil and water viscosities are equal. Therefore, the oil fractional curves $f(S)$ result

$$f(S) = \frac{S^2}{0.2\,(1-S)^2 + S^2} \tag{38}$$

and

$$f(S) = \frac{S^2}{0.2\,(1-S) + S^2}. \tag{39}$$

The fractional flow is obtained numerically, integrating the estimated function.

For the numerical tests the observations $S^{obs}(t_j)$ and $P^{obs}(t_j)$ are the evaluation of the exact solution at $x = 0.98$ and times $t_j = j * 0.05$, $j = 1, \ldots, 20$. The accuracy required for convergence is TOL$= 10^{-10}$.

Figure 1 displays the exact solution H, the initial guess H^0, the first iteration H^1 and the optimum estimated H^k for example 1. Convergence was achieved in nine iterations.

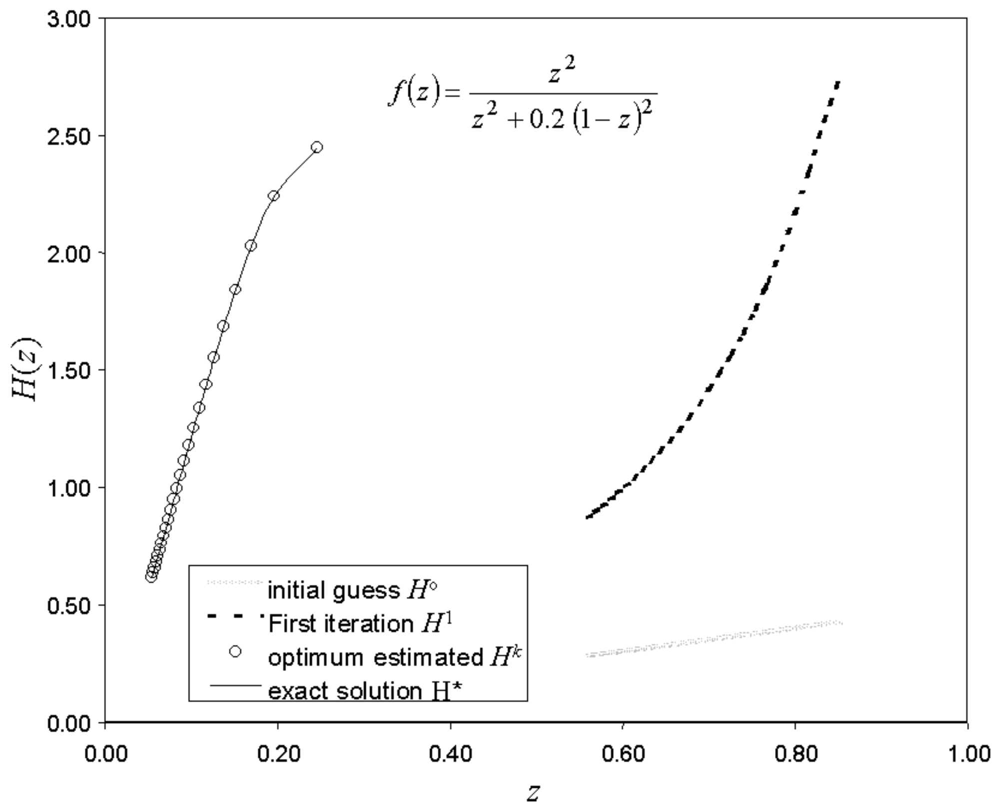

Figure 1: Estimation of the derivative of the fractional flow curve for example 1

The approximation of the inverse of the total mobility for example 1 is shown in the following figure:

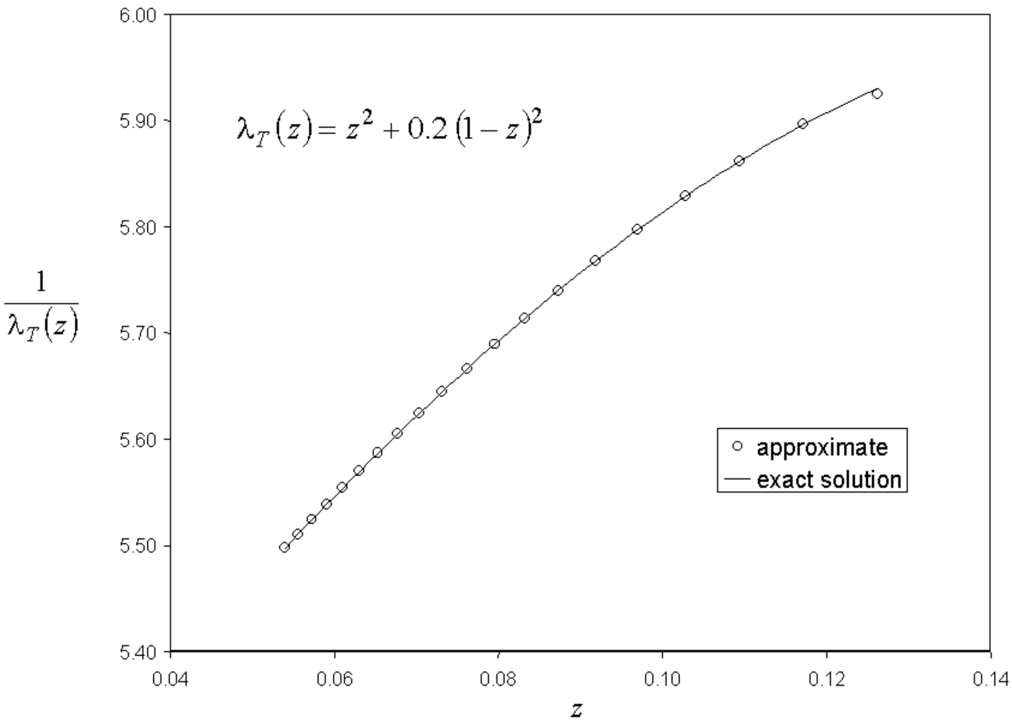

Figure 2: Estimation of the inverse of the total mobility for example 1

Similarly Figures 3 and 4 show the estimation of the H and $1/\lambda_T$ for example 2. In this case convergence for the parameter H was achieved in 12 iterations.

An almost exact match to the 'true' function is obtained in every case.

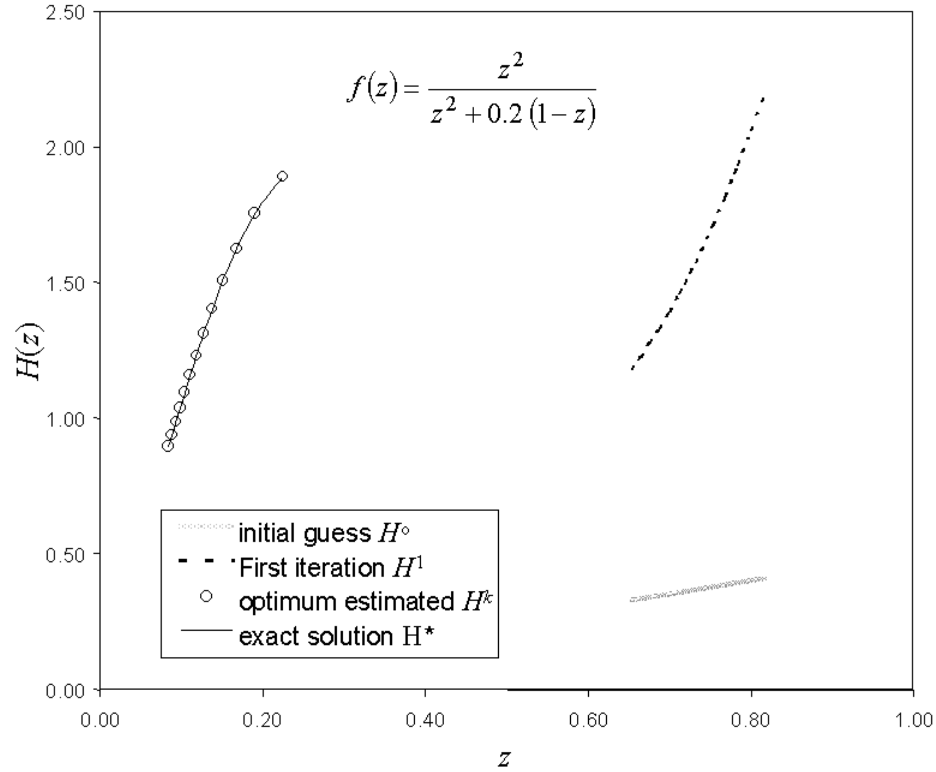

Figure 3: Estimation of the derivative of the fractional flow curve for example 2

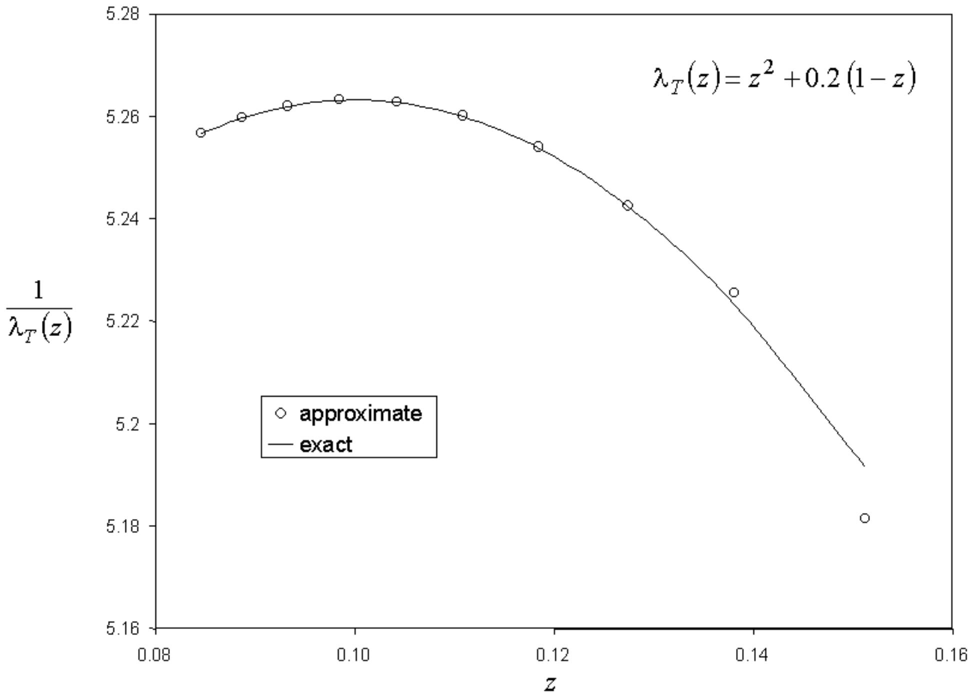

Figure 4: Estimation of the inverse of the total mobility for example 2

From these examples and some other cases tested, the algorithm proved numerically successful.

6 CONCLUSIONS

An algorithm to solve an estimation problem in a non-linear system is introduced. This algorithm is based on the least squares criterion in functional spaces and therefore an infinite dimensional estimation problem is solved. The sensitivity equation that is used to compute the Frèchet derivative is presented. The resulting algorithm has shown convergence in numerical tests, and because of its general theoretical formulation has the potential to be extended to solve more complex problems. This algorithm obtains the sought-after parameters without the imposition of a priori parametric models. Though the use of such models is currently the common practice among field engineers, different models yield different results and there is no objective criterion to choose among them. Thus the result yielded by the different parametric models can be compared to those of our algorithm, so as to select the most fitting model.

7 APPENDIX

We show the steps that lead to Eq. (24) starting from its continuous version (20) and using the discrete versions of H and $S(H)$.

We denote by Ψ the subespace of $L^2([z_1^k, z_M^k])$ that is generated by the set of functions $\{\psi_j^k\}_{j=1\ldots,M}$. From Eq.(22) $H^k \in \Psi$. From Eq.(24) $S(H^k)(x^{rec}, t) \in L^2([0,T])$, therefore, $S'_H(x^{rec}, \cdot) \in L^2([0,T])$, specifically,

$$\begin{aligned} S'_H &: \Psi \to L^2([0,T]), \\ \delta H(z) &\to S'_H \delta H(x^{rec}, t), \end{aligned} \tag{40}$$

as a consequence,

$$\left(S'_H\right)^* : L^2([0,T]) \to \Psi. \tag{41}$$

To compute the coefficients $\delta\gamma_j^k$ of δH we proceed as usual. We calculate the scalar product of (20) with an element of the basis of Ψ.

The computation of the left hand side, for $j = 1, \ldots, M$ results:

$$\begin{aligned} &\left(S'_H(S(H^k))^* S'_H(S(H^k))\delta H \,,\, \psi_j^k\right)_{L^2([z_1^k, z_M^k])} \\ &= \left(S'_H(S(H^k))\delta H \,,\, S'_H\left(S(H^k)\right)\psi_j^k\right)_{(L^2[0,T])}. \end{aligned} \tag{42}$$

Using Eq.(18) to replace $S'_H(S(H^k))$ in the above equation and writing explicitly the scalar product in $L^2([0,T])$, Eq.(42) equals to

$$\int_0^T \left({S'_H}^2\, \delta H\left(S(H^k)\right) \psi_j^k\left(S(H^k)\right)\right)(x^{rec}, t)\, dt. \tag{43}$$

Discretizing Eq.(43) in t, using the definition of ψ_j^k and that of z_j^k (Eq.(26)), we obtain the chain of equalities:

$$\begin{aligned} &\sum_{l=1}^{M} \left({S'_H}^2\, \delta H\left(S(H^k)\right) \psi_j^k\left(S(H^k)\right)\right)(x^{rec}, t_l)\, \Delta t \\ &= \left({S'_H}^2\, \delta H\left(S(H^k)\right)(x^{rec}, t_{M-j+1}\right) \Delta t \\ &= {S'_H}^2(x^{rec}, t_{M-j+1})\, \delta\gamma_j^k\, \Delta t \end{aligned} \tag{44}$$

Similarly, the right hand side of (20) is,

$$
\begin{aligned}
&-\int_0^T \left(S(H^k) - S^{obs}\right)\Big(S'_H\ \psi_j^k\left(S(H^k)\right)\Big)(x^{rec},t)\,dt \\
&\approx -\sum_{l=1}^{M} \left(S(H^k) - S^{obs}\right)\left(S'_H\,\psi_j^k\left(S(H^k)\right)(x^{rec},t_l)\,\Delta t\right. \\
&\qquad = -\Big(\left(S(H^k) - S^{obs}\right)S'_H\Big)(x^{rec},t_{M-j+1})\,\Delta t.
\end{aligned}
\tag{45}
$$

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