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THE COUPLING OF THE BIOLOGICAL GROWTH METHOD AND THE BOUNDARY ELEMENT METHOD FOR STRUCTURAL SHAPE OPTIMISATION

C. Wessel [a,b], L. C. Wrobel [a] and A. P. Cisilino [b]

[a] *School of Engineering and Design, Brunel University, Uxbridge, Middlesex, UK (UB8 3PH)*
cwessel@fi.mdp.edu.ar
[b] *Facultad de Ingeniería, Universidad Nacional de Mar del Plata – CONICET. Av. Juan B. Justo 4302 Mar del Plata, Argentina (7600*

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Abstract. A numerical evolutionary procedure for structural optimisation of two-dimensional structures based on the Biological Growth Method (BGM) is shown and is implemented using two Boundary Element Method (BEM) formulations: the standard for two-dimensional elastostatics for the stress or strain analysis, and the Dual Reciprocity Method (DRM) for modelling the swelling/shrinking of the optimisation domain. Advancing previous work done by the authors on the original stress formulation of the BGM, tangential strains are used as objective functions in this paper.

1 INTRODUCTION

Biological Growth Method. Mattheck (1990) introduced the Biological Growth Method (BGM) based on his observations in Nature. The BGM major hypothesis is that the process of optimisation in natural structures is carried out through the swelling or shrinking of the soft outermost layer of material, which yields the levelling of local stresses. On the other hand, Mattheck defines *optimum shape* as the one that shows a state of constant stress at part of, or the whole of, the surface of the component. Consequently, optimisation can be described as the minimization of the equation

$$\dot{\varepsilon}_v = k\left(\sigma_{vm} - \sigma_{ref}\right) \tag{1}$$

where $\dot{\varepsilon}_v$ is the volumetric swelling strain rate, σ_{vm} is the von Mises stress and σ_{ref} is a reference stress, an expected value. This equation holds for each point in the optimisation domain.

An elegant method to implement eq. (1) is by means of a thermal expansion analogy based on the generalized Hooke's law, which gives:

$$\alpha\theta = \gamma k\left(\sigma_{vm} - \sigma_{ref}\right) \tag{2}$$

where θ is the temperature field, α is the thermal expansion coefficient and γ is a units conversion factor.

The present authors (Wessel *et. al*, 2004 and 2005) have implemented eq. (2) in previous works. Given the fact that the BGM is a heuristic method that only considers physical variables, and also that in some fields strains are required as objective functions (as will be discussed later), the present work makes use of the following equation instead of eq. (2):

$$\alpha\theta = \gamma k\left(\varepsilon_{tan} - \varepsilon_{ref}\right) \tag{3}$$

where ε_{tan} and ε_{ref} are the tangential strain and a reference tangential strain, respectively.

Eq. (2) has been previously implemented in Finite Element Method (FEM) codes (Tekkaya and Guneri, 1995 and Li *et. al*. 1999). However, the fact that optimisation takes place only in the outermost layer in BGM makes an implementation with the Boundary Element Method (BEM) computationally cheaper. The boundary-only intrinsic characteristic of the BEM together with its accuracy in the boundary displacement and stress or strains solutions justifies the use of the BEM in this work.

BEM for two-dimensional elasticity. The starting point of the BEM formulation for two-dimensional elasticity is the Navier equation

$$Gu_{j,kk} + \frac{G}{1-2\nu}u_{k,kj} + b_j = 0 \tag{4}$$

where j,k denote Cartesian components, G is the shear modulus, ν is the Poisson's ratio, b_j are the components of body forces and u_k are the displacements.

Following Brebbia *et al.* (1984) the corresponding boundary integral equation for a domain Ω confined by the boundary surface Γ is:

$$c^i_{lk}(x')u^i_k(x') + \int_\Gamma p^*_{lk}(x',x)u_k(x)d\Gamma(x) = \int_\Gamma u^*_{lk}(x',x)p_k(x)d\Gamma(x) + \int_\Omega u^*_{lk}(x',x)b_k(x)d\Omega(x) \quad (5)$$

where $u^*_{lk}(x',x)$ is the fundamental solution of eq. (4) and $p^*_{lk}(x',x)$ its corresponding traction; u_k and p_k are the displacements and tractions in the boundary Γ, respectively, and $c_{lk}(x')$ is a jump term related to the boundary geometry.

If there are no body forces present, eq. (5) is reduced to the boundary-only equation

$$c^i_{lk}(x')u^i_k(x') + \int_\Gamma p^*_{lk}(x',x)u_k(x)d\Gamma(x) = \int_\Gamma u^*_{lk}(x',x)p_k(x)d\Gamma(x) \quad (6)$$

The basic idea behind the BEM is to solve eq. (6) numerically. To accomplish this, the model contour is discretized into N elements, where displacements $u_k(x)$ and tractions $p_k(x)$ are expressed in terms of the nodal values u^i_k and p^i_k by means of isoparametric interpolation functions. This process results in an algebraic system of equations from which the unknown nodal values of u^i_k and p^i_k can be recovered.

It should be noticed that eq. (6) only involves integrals on Γ. Consequently, a typical BEM formulation only requires a boundary discretization.

The Dual Reciprocity BEM (DRM) for two-dimensional thermoelasticity. Thermal effects (as much as body forces) were initially a restriction in the use of BEM as they must be included in the formulation by means of a domain integral (see eq. (5)), thus losing the advantage of the method's "boundary-only" character. Many different approaches have been developed to overcome this problem, among which DRM has become widely used. The basic idea behind this approach is to employ fundamental solutions and global approximation functions, as described in what follows.

Following Partridge and Sensale (1997) the effects produced by changes in temperature θ in elastic bodies can be represented by initial stresses σ^0_{jk} such that:

$$\sigma^0_{jk} = \chi\theta\delta_{jk} \quad (7)$$

where $\chi = -2G\frac{1+\nu}{1-2\nu}\alpha$, so that eq. (5) becomes

$$c^i_{lk}(x')u^i_k(x') + \int_\Gamma p^*_{lk}(x',x)u_k(x)d\Gamma(x) = \int_\Gamma u^*_{lk}(x',x)p_k(x)d\Gamma(x) + \int_\Omega \chi\theta_{,\kappa}(x)u^*_{lk}(x',x)d\Omega - \int_\Gamma \chi u^*_{lk}(x',x)\theta(x)n_k(x)d\Gamma(x) \quad (8)$$

In DRM changes in the temperature θ are expressed in terms of known co-ordinate functions f^j, which are also temperature fields:

$$\theta \approx \sum_{j=1}^{N+L+A} f^j \beta^j \ , \tag{9}$$

where β^j is a set of initially unknown coefficients. N points are placed on the contour and L in the domain, and A augmentation functions are used to improve the approximation.

Next, a particular solution $\hat{u}^j_{mk}$ to eq. (4) corresponding to the generic function f^j is found. Then, replacing eq. (9) into (8) results in a boundary-only equation:

$$\begin{aligned} & c^i_{lk}(x')u^i_k(x') + \int_\Gamma p^*_{lk}(x',x)u_k(x)d\Gamma(x) - \\ & \int_\Gamma u^*_{lk}(p_k - \chi\theta n_k)d\Gamma(x) = \\ & = \sum_{j=1}^{N+L+A}\left[c^i_{lk}\hat{u}^{ij}_{mk} + \int_\Gamma p^*_{lk}\hat{u}^j_{mk}d\Gamma - \int_\Gamma u^*_{lk}(\hat{p}^j_{mk} - \chi f_j n_{mk})d\Gamma\right] \end{aligned} \tag{10}$$

where $\hat{p}^j_{mk}$ are the particular traction solutions corresponding to particular displacements $\hat{u}^j_{mk}$. The procedure for numerical solution of eq. (10) follows that described for eq. (6).

The choice of approximation functions in eq. (9) is somewhat arbitrary. Generally a radial basis function is used, such as thin plate splines or multiquadrics. These have shown to interpolate only in the neighbourhood of a particular point (local behaviour), so that global functions are also needed. For these, terms in the Pascal triangle or global sine Pascal triangle are often employed.

2 IMPLEMENTATION

The process described by eq. (3) was implemented using a standard BEM and a multi-region DRM formulations in the following manner:

i. An appropriate BEM discretisation is generated for the original (and subsequent) models using quadratic isoparametric elements. In addition to the boundary nodes, internal collocation points are set.
ii. Tangential strains are computed on the model boundary nodes and on all of the internal points by means of the standard formulation of BEM.
iii. A thermal expansion analysis is performed using the DRM formulation with a temperature field θ given by eq. (3). In order to limit the swelling to the outermost layer of material, a non-zero temperature field is specified only on the optimisation boundary nodes and the optimisation internal points. In this work, the thin plate spline $r^2 log(r)$ was applied and terms up to the second degree in the Pascal (TAPT3 combination) triangle were chosen as augmentation functions (Bridges and Wrobel, 1996 and Partridge and Sensale, 1997).
iv. The optimisation boundary geometry is smoothed using local cubic splines for mid-point relocation (see Das Bhaumik, 2005).

Steps ii to iv are repeated until acceptably low values of (ε_{tan} - ε_{ref}) are obtained.

3 EXAMPLE

The problem presented herein consists of a structure of bone tissue which may grow into a slot, as shown in Fig. 1. This problem was also solved by Sadegh *et al.*(1993).

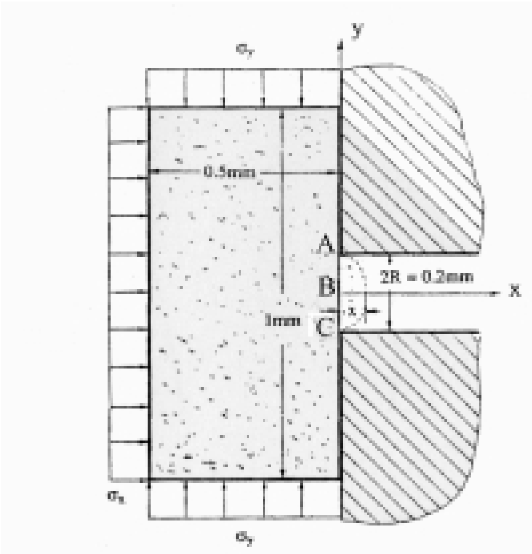

Figure 1. Bone tissue structure subject to tensions σ_x y σ_y, that may grow into a slot.

Models for the architectural changes that occur in bone tissue around or near an implant are developed based on the assumption that bony changes are an adaptive response and that bone surface strains control this adaptive response.

For example, Lanyon and coworkers (1984 and 1987) have evaluated the relationship between bone tissue response and tissue level peak strain magnitude in strain-gauged animal experiments for a number of species. They have shown that bone resorption occurs for tissue-level peak compressive strains less than about 0.001, and bone deposition occurs for tissue-level peak compressive strains greater than about 0.003. For this reason we have set ε_{ref} =-0.002 in this work.

The selected material properties are a Young's modulus of 17.12 GPa and a Poisson's ratio of 0.28, while σ_x=8 MPa and σ_y=80 MPa. Only one half of the structure was considered for symmetry reasons. The optimization domain, limited by BC and with a surface of 0.001 mm^2, includes 26 quadratic elements.

During the optimization process bone tissue grows into the slot (for these values of applied stresses), lowering the difference (ε_{tan} - ε_{ref}). This can be seen in Fig 2, where the initial and final configurations of the optimisation domain are shown.

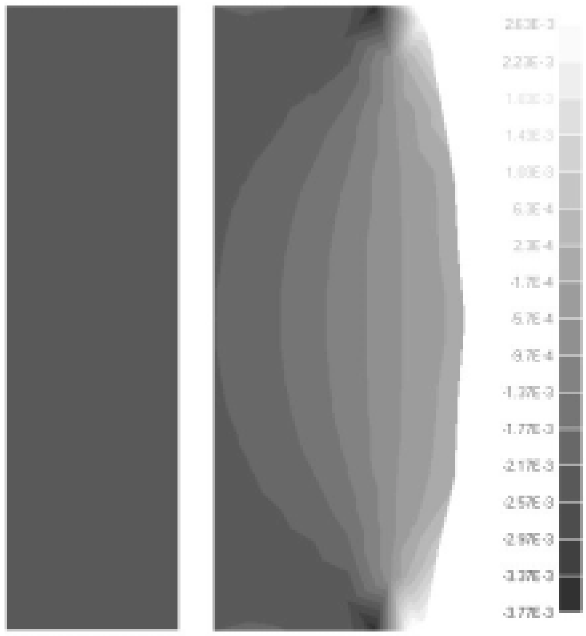

Figure 2. Initial and final geometry of the optimization domain and their corresponding values of (ε_{tan} - ε_{ref})

Finally, Fig. 3 and 4 show the smooth evolution of the position and difference (ε_{tan} - ε_{ref}) for point B

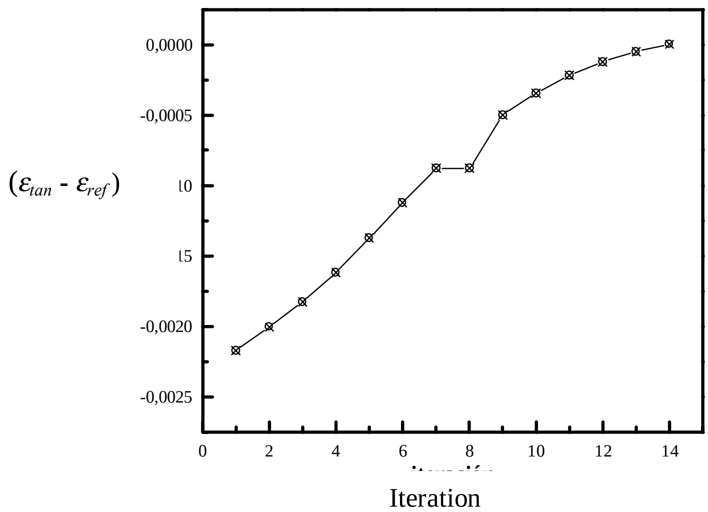

Figure 3. Evolution of (ε_{tan} - ε_{ref}) for point B during the optimisation process.

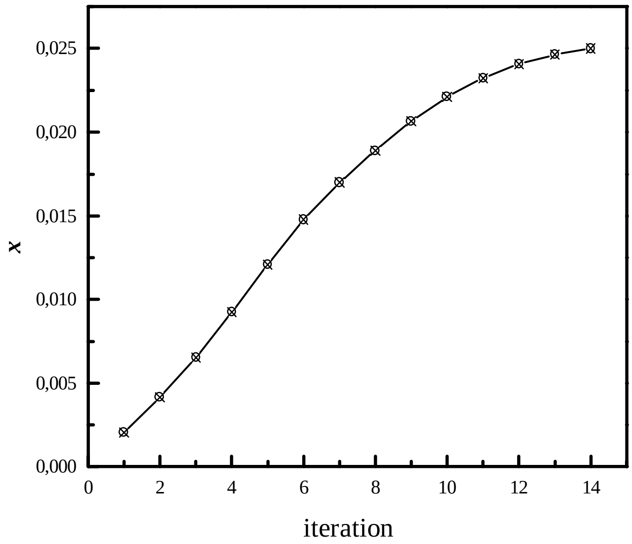

Figure 4. Position of point B during the optimisation process.

Fig. 4 shows the smooth evolution of the position of point B to a value of 0.0250 mm, in complete agreement with the results reported by Sadegh *et al.*(1993).

4 CONCLUSIONS

A novel numerical evolutionary procedure for the structural shape optimisation of two-dimensional problems based on the Biological Growth Method was presented in this work. The versatility of the proposed methodology has been illustrated with one example that shows complete agreement with existing results.

Results obtained using von Mises stresses as objective functions (not presented herein) also show excellent agreement with analytical results. The change of objective function is simple to implement in the programme developed by the authors.

Finally, the implementation using boundary elements makes the algorithm simple and computationally inexpensive.

5 ACKNOWLEDGEMENTS

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