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REVISIÓN DE MODELOS DE CONDUCCIÓN DEL CALOR Y DE DIFUSIÓN EN SÓLIDOS MICROESTRUCTURADOS Y EN METAMATERIALES II: SOLUCIONES SEMIANALITICAS

REVIEW OF HEAT CONDUCTION AND DIFFUSION MODELS IN MICROSTRUCTURED SOLIDS AND METAMATERIALS II: SEMI-ANALYTICAL SOLUTIONS

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Palabras clave: Termoelasticidad, poro-termo-elasticidad, micro-temperaturas, teoría de segundo gradiente.

Resumen. En el presente trabajo se escriben detalladamente los sistemas de ecuaciones de reacción difusión de tipo Onsager, involucrados en los procesos de transferencia de calor y masa en medios microestructurados (S. Forest y E.C. Aifantis, *Int J Solids Struct*, 47(25):3367–3376 (2010)), descriptos en la síntesis anterior, además de sus condiciones iniciales y de borde respectivas. Luego se definen las funciones de Green, posteriormente, utilizando los conocidos teoremas de representación de Green Lagrange, se construyen sistemas de soluciones representadas por ecuaciones integrales acopladas genéricas, las cuales se resuelven, utilizando la técnica de aproximantes de Picard, formulándose, un método sistemático para el cálculo perturbativo de las soluciones (D. Jou y V. Cimmelli, *Commun Appl Ind Math*, 7 (2016)).

Keywords: Thermoelasticity, poro-thermo-elasticity, micro-temperatures, second gradient theory.

Abstract. In the present work, the systems of Onsager-type fusion reaction equations, involved in the processes of heat and mass transfer in microstructured media (S. Forest and E.C. Aifantis, *Int J Solids Struct*, 47(25):3367–3376 (2010)), described in the previous synthesis, as well as their respective initial and edge conditions, are written in detail. Then, Green's functions are defined, then, using the well-known Green Lagrange representation theorems, solution systems represented by generic coupled integral equations are constructed, which are solved, using Picard's approximant technique, formulating a systematic method for the perturbative calculation of solutions (D. Jou and V. Cimmelli, *Commun Appl Ind Math*, 7 (2016)).

1. INTRODUCCIÓN

A partir de algunas de las estructuras teóricas desarrolladas en el trabajo anterior, se propone en este, la formulación del problema de condiciones iniciales y de borde de dos modelos:

a) **Modelo quimio-poroelástico cuasi-lineal** de Aifantis (2016), en la formulación de segundo gradiente referido a un medio poroso, con dos porosidades y temperaturas, las variables descriptoras son:

$$\Re = \{u_j, l_1^2\hat{\nabla}^2 u_j, c, l_1^2\hat{\nabla}^2 c, p_a, p_b, T\} \ ; \ \partial M = \partial\Gamma_1 \cup \partial\Gamma_2$$

b) **Modelo termo-elastodinámico con micro-temperaturas** en la concepción de Ieşan y Scalia (2014), acoplado a un campo de temperaturas en la formulación de Green Naghdi tipo II, las variables descriptoras son:

$$\Re = \{u_j, l_1^2\hat{\nabla}^2 u_j, w_j, T\} \ ; \ \partial M = \partial\Gamma_1 \cup \partial\Gamma_2$$

2. FORMULACIÓN DEL PROBLEMA DE CONDICIONES INICIALES Y DE BORDE PARA LA QUIMIO-PORO-ELASTODINÁMICA ACOPLADA A UN CAMPO DE TEMPERATURAS DE TIPO GREEN-NAGHDI TIPO II

Distribución de campos de desplazamiento en la formulación de segundo gradiente

$$\begin{aligned}
&\rho\partial_t^2 u_j(\vec{x},t) - \rho l_1^2(\partial_t^2 u_j(\vec{x},t))_{,kk} - \mu(1 - l_1^2\hat{\nabla}^2)\hat{\nabla}^2 u_j(\vec{x},t) - \\
&\quad - (\mu+\lambda)\hat{\nabla}((1 - l_1^2\hat{\nabla}^2)(\hat{\nabla} u_k(\vec{x},t))) + \xi \boldsymbol{M}_{jk}(c(\vec{x},t) - l_1^2\hat{\nabla}^2 c(\vec{x},t))_{,k} + \\
&\quad\quad + \alpha_1 \boldsymbol{M}_{jk}^a(1 - l_1^2\hat{\nabla}^2)p_{a,k}(\vec{x},t) + \alpha_2 \boldsymbol{M}_{jk}^b(1 - l_1^2\hat{\nabla}^2)p_{b,k}(\vec{x},t) + \\
&\quad\quad\quad + \beta \boldsymbol{M}_{jk}^E(1 - l_1^2\hat{\nabla}^2)(1+\partial_t)T_k(\vec{x},t) = -\boldsymbol{S}_{jklm}^{Es}\boldsymbol{\varepsilon}_{lm,k}^*(\vec{x},t) \quad \text{en } R_k \\
&u_j(\vec{x},0) = u_j^0 \ / \ u_j^0 \in (\boldsymbol{H}_0^1(D_k))^3 \ ; \ \partial_t u_j(\vec{x},0) = w_j^0 \ / \ w_j^0 \in (\boldsymbol{L}^2(D_k))^3
\end{aligned}$$

Distribución de concentraciones en la formulación de segundo gradiente

$$\begin{aligned}
&\partial_t c(\vec{x},t) - g_2\hat{\nabla}^2 c(\vec{x},t) + g_3\hat{\nabla}^4 c(\vec{x},t) + g_6(\hat{\nabla}^2(\hat{\nabla} u_k(\vec{x},t))) - l_1^2\hat{\nabla}^4(\hat{\nabla} u_k(\vec{x},t)) + \\
&\quad + \xi_1\partial_t\mathcal{H}\{T, p_a, p_b\} = 0 \quad \text{en } R_k \\
&c(\vec{x},0) = c_0 \ / \ c_0 \in H_0^1(D_k)
\end{aligned}$$

Distribuciones de presiones de poro

$$\begin{aligned}
&\partial_t p_a(\vec{x},t) - D_a^2\hat{\nabla}^2 p_a(\vec{x},t) + \bar{\alpha}_1\boldsymbol{M}_{jk}^a(1 - l_1^2\hat{\nabla}^2)\dot{u}_{j,k}(\vec{x},t) + \xi_1\partial_t\mathcal{H}\{T, p_b\} = 0 \quad \text{en } R_k \\
&\partial_t p_b(\vec{x},t) - D_b^2\hat{\nabla}^2 p_b(\vec{x},t) + \bar{\alpha}_2\boldsymbol{M}_{jk}^b(1 - l_1^2\hat{\nabla}^2)\dot{u}_{j,k}(\vec{x},t) + \xi_2\partial_t\mathcal{H}\{T, p_a\} = 0 \quad \text{en } R_k \\
&p_a(\vec{x},0) = p_a^0 \ / \ p_a^0 \in H_0^1(D_k) \ ; \ p_b(\vec{x},0) = p_b^0 \ / \ p_b^0 \in H_0^1(D_k)
\end{aligned}$$

Distribuciones de temperaturas en la formulación de Lord-Shulman

$$\begin{aligned}
&\rho c_v\tau\partial_t^2 T(\vec{x},t) - (\boldsymbol{K}_{jk}^E T_{,j}(\vec{x},t))_{,k} + \rho c_v\tau\partial_t T(\vec{x},t) + \\
&\quad + T_0\bar{\beta}\boldsymbol{M}_{jk}^E(1 - l_1^2\hat{\nabla}^2)(\dot{u}_{j,k}(x,t) + \tau\ddot{u}_{j,k}(\vec{x},t)) + \xi_1\partial_t\mathcal{H}\{p_a, p_b\} = 0 \quad \text{en } R_k \\
&T(\vec{x},0) = T_0 \ / \ T_0 \in H_0^1(D_k) \ ; \ \partial_t T(\vec{x},0) = q_0 \ / \ q_0 \in L^2(D_k)
\end{aligned}$$

Condiciones de borde mixtas

$$
\begin{aligned}
&(-\mu\hat{n}_k u_{j,k}(\vec{x},t) + l_1^2\hat{n}_k\hat{\nabla}^2 u_{j,k}(\vec{x},t))\Big|_{\partial\Gamma_1} - \\
&\quad - (\mu+\lambda)\hat{n}_j((1-l_1^2\hat{\nabla}^2)(u_{k,k}(\vec{x},t)))\Big|_{\partial\Gamma_1} + \xi\boldsymbol{M}_{jk}(c(\vec{x},t) - l_1^2\hat{\nabla}^2 c(\vec{x},t))\hat{n}_k\Big|_{\partial\Gamma_1} + \\
&\qquad + (\alpha_1\boldsymbol{M}^a_{jk}(1-l_1^2\hat{\nabla}^2)p_a(\vec{x},t) + \alpha_2\boldsymbol{M}^b_{jk}(1-l_1^2\hat{\nabla}^2)p_b(\vec{x},t))\hat{n}_k\Big|_{\partial\Gamma_1} + \\
&\qquad\quad + \beta\boldsymbol{M}^E_{jk}(1-l_1^2\hat{\nabla}^2)(1+\partial_t)T(\vec{x},t)\hat{n}_k\Big|_{\partial\Gamma_1} = q^a_j\Big|_{\partial\Gamma_1} /\ q^a_j \in (\boldsymbol{L}^2(\partial\Gamma_1)) \\
&(-g_2 c(\vec{x},t)_{,k} + g_3\hat{\nabla}^2 c(\vec{x},t)_{,k})\hat{n}_k\Big|_{\partial\Gamma_1} + g_6((\hat{\nabla}u_k(\vec{x},t)) - l_1^2\hat{\nabla}^4(\hat{\nabla}u_k(\vec{x},t))_{,k})\hat{n}_k\Big|_{\partial\Gamma_1} = 0 \\
&(-D_A^2 p_{a,k}(\vec{x},t) + \bar{\alpha}_1\boldsymbol{M}^a_{jk}(1-l_1^2\hat{\nabla}^2)\dot{u}_j(\vec{x},t))\hat{n}_k\Big|_{\partial\Gamma_1} = 0 \\
&(-D_B^2 p_{b,k}(\vec{x},t) + \bar{\alpha}_2\boldsymbol{M}^b_{jk}(1-l_1^2\hat{\nabla}^2)\dot{u}_j(\vec{x},t))\hat{n}_k\Big|_{\partial\Gamma_1} = 0 \\
&(-\boldsymbol{K}^E_{jk}T_{,j}(\vec{x},t) + T_0\bar{\beta}\boldsymbol{M}^E_{jk}(1-l_1^2\hat{\nabla}^2)(\dot{u}_j(\vec{x},t) + \tau\ddot{u}_j(\vec{x},t)))\hat{n}_k\Big|_{\partial\Gamma_1} = 0 \\
&c(\vec{x},t)\Big|_{\partial\Gamma_2} = u_j(\vec{x},t)\Big|_{\partial\Gamma_2} = 0\ ;\ p_a(\vec{x},t)\Big|_{\partial\Gamma_2} = p_b(\vec{x},t)\Big|_{\partial\Gamma_2} = 0\ ;\ T(\vec{x},t)\Big|_{\partial\Gamma_2} = 0
\end{aligned}
$$

Las constantes que intervienen en el modelo son todas reales positivas, las matrices asociadas a los tensores de segundo orden son simétricas y sus formas cuadráticas propias son definidas positivas. Las funciones $\mathcal{H}$ se definen como funciones de tipo histeretico esencialmente de tipo Preisach. (Cacciola et al., 2009) (Mörée y Leijon, 2023)

$\boldsymbol{S}^{Es}_{jklm}$ es el tensor de Eshelby-Mura, que satisface

$$\boldsymbol{S}^{Es}_{jklm} = \boldsymbol{S}^{Es}_{kjlm} = \boldsymbol{S}^{Es}_{jkml}$$

$\boldsymbol{\varepsilon}_{lm,k} \in (\boldsymbol{L}^2(R_k))^{3\times 3}$: Deformaciones residuales.

2.1. Definición de las funciones de Green

$$
\begin{aligned}
&-\partial_t g_c(\Delta\vec{x},\Delta t) - g_2\hat{\nabla}^2 g_c(\Delta\vec{x},\Delta t) + g_3\hat{\nabla}^4 g_c(\Delta\vec{x},\Delta t) = \delta(\Delta\vec{x},\Delta t) \\
&-\partial_t g_a(\Delta\vec{x},\Delta t) - D_A^2\hat{\nabla}^2 g_a(\Delta\vec{x},\Delta t) = \delta(\Delta\vec{x},\Delta t) \\
&-\partial_t g_b(\Delta\vec{x},\Delta t) - D_B^2\hat{\nabla}^2 g_b(\Delta\vec{x},\Delta t) = \delta(\Delta\vec{x},\Delta t) \\
&\rho c_v\tau\partial_t^2 g_T(\Delta\vec{x},\Delta t) - (\boldsymbol{K}^E_{jk} g_T(\Delta\vec{x},\Delta t)_{,j})_{,k} + \rho c_v\tau\partial_t g_T(\Delta\vec{x},\Delta t) = \delta(\Delta\vec{x},\Delta t) \\
&\rho\partial_t^2 g_{jk}(\Delta\vec{x},\Delta t) - \mu(1-l_1^2\hat{\nabla}^2)\hat{\nabla}^2 g_{jk}(\Delta\vec{x},\Delta t) - \\
&\qquad\qquad - (\mu+\lambda)((1-l_1^2\hat{\nabla}^2)(g_{jn,n}(\Delta\vec{x},\Delta t)))_{,k} = \delta_{jk}\delta(\Delta\vec{x},\Delta t) \\
&g_c(\Delta\vec{x},-t_f) = g_a(\Delta\vec{x},-t_f) = g_b(\Delta\vec{x},-t_f) = 0 \\
&g_T(\Delta\vec{x},0) = \partial_t g_T(\Delta\vec{x},0) = g_{jk}(\Delta\vec{x},0) = \partial_t g_{jk}(\Delta\vec{x},0) = 0
\end{aligned}
$$

Condiciones de borde

$$(-\mu\hat{n}_l g_{jk,l}(\Delta\vec{x},\Delta t) + l_1^2\hat{n}_l\hat{\nabla}^2 g_{jk,l}(\Delta\vec{x},\Delta t))\Big|_{\partial\Gamma_1} - \\ -(\mu+\lambda)\hat{n}_j((1-l_1^2\hat{\nabla}^2)(g_{kn,n}(\Delta\vec{x},\Delta t)))\Big|_{\partial\Gamma_1} = 0$$

$$(-g_2 g_c(\Delta\vec{x},\Delta t)_{,k} + g_3\hat{\nabla}^2 g_c(\Delta\vec{x},\Delta t)_{,k,k})\hat{n}_k\Big|_{\partial\Gamma_1} = 0\ ;\ g_T(\Delta\vec{x},\Delta t)\Big|_{\partial\Gamma_2} = 0$$

$$-D_A^2 g_{a,k}(\Delta\vec{x},\Delta t)\hat{n}_k\Big|_{\partial\Gamma_1} = -D_B^2 p_{b,k}(\vec{x},t)\hat{n}_k\Big|_{\partial\Gamma_1} = -\boldsymbol{K}_{jk}^E g_{T,j}(\Delta\vec{x},\Delta t)\hat{n}_k\Big|_{\partial\Gamma_1} = 0$$

$$g_c(\Delta\vec{x},\Delta t)\Big|_{\partial\Gamma_2} = g_{jk}(\Delta\vec{x},\Delta t)\Big|_{\partial\Gamma_2} = 0\ ;\ g_a(\Delta\vec{x},\Delta t)\Big|_{\partial\Gamma_2} = g_b(\Delta\vec{x},\Delta t)\Big|_{\partial\Gamma_2} = 0$$

2.2. Construcción de las soluciones semianaliticas

Utilizando el segundo y el tercer teorema de representación de Green-Lagrange podemos construir las representaciones integrales de las soluciones en la forma siguiente:

$$u_j(\vec{x},t) = \iiint_{V_k} d^3x'\{g_{jk}(\vec{x}-\vec{x}',t)\partial_t u_k(\vec{x}',0) - u_k(\vec{x}',0)\partial_t g_{jk}(\vec{x}-\vec{x}',t)\} + \\ + \int_0^t dt'\left\{\iiint_{V_k} d^3x' g_{jk}(\vec{x}-\vec{x}',t-t')\Xi_k(\vec{x}',t',\wp(u_j,p_a,p_b,c,T))\right\} + \\ + \int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_{jk}(\vec{x}-\vec{x}',t-t')\left\{q_j^a(\vec{x}',t')\Big|_{\partial\Gamma_1} + \wp_j^s(u_j,p_a,p_b,T,c)\Big|_{\partial\Gamma_1}\right\}\right\}\right\}$$

$$c(\vec{x},t) = \iiint_{V_k} d^3x'\{g_c(\vec{x}-\vec{x}',t)c(\vec{x}',0)\} + \\ + \int_0^t dt'\left\{\iiint_{V_k} d^3x' g_c(\vec{x}-\vec{x}',t-t')\Xi_c(\vec{x}',t',\wp(u_j,c))\right\} + \\ + \int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_c(\vec{x}-\vec{x}',t-t')\wp_c^s(u_j,T)\Big|_{\partial\Gamma_1}\right\}\right\}$$

$$p_a(\vec{x},t) = \iiint_{V_k} d^3x'\{g_a(\vec{x}-\vec{x}',t)p_a(\vec{x}',0)\} + \\ + \int_0^t dt'\left\{\iiint_{V_k} d^3x' g_a(\vec{x}-\vec{x}',t-t')\Xi_a(\vec{x}',t',\wp(u_j,p_a,p_b,c,T))\right\} + \\ + \int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_a(\vec{x}-\vec{x}',t-t')\wp_a^s(u_j,p_a,p_b,T)\Big|_{\partial\Gamma_1}\right\}\right\}$$

$$p_b(\vec{x},t) = \iiint_{V_k} d^3x'\{g_b(\vec{x}-\vec{x}',t)p_b(\vec{x}',0)\} + \\ + \int_0^t dt'\left\{\iiint_{V_k} d^3x' g_b(\vec{x}-\vec{x}',t-t')\Xi_b(\vec{x}',t',\wp(u_j,p_a,p_b,c,T))\right\} + \\ + \int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_b(\vec{x}-\vec{x}',t-t')\wp_b^s(u_j,p_a,p_b,T)\Big|_{\partial\Gamma_1}\right\}\right\}$$

$$T(\vec{x},t) = \iiint_{V_k} d^3x'\{g_T(\vec{x}-\vec{x}',t)\partial_t T(\vec{x}',0) - T(\vec{x}',0)\partial_t g_T(\vec{x}-\vec{x}',t)\}+$$
$$+\int_0^t dt'\left\{\iiint_{V_k} d^3x' g_{jk}(\vec{x}-\vec{x}',t-t')\Xi_T(\vec{x}',t',\wp_T(u_j,p_a,p_b,c,T))\right\}+$$
$$+\int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_T(\vec{x}-\vec{x}',t-t')\wp_T^s(u_j,p_a,p_b,T)\Big|_{\partial\Gamma_1}\right\}\right\}$$

2.3. Construcción de los aproximantes de Picard

Los aproximantes de Picard serán:

$$u_j^{(n+1)}(\vec{x},t) \cong K_j^0(u_j^0,w_j^0)+$$
$$+\int_0^t dt'\left\{\iiint_{V_k} d^3x' g_{jk}(\vec{x}-\vec{x}',t-t')\Xi_k^{(n)}(\vec{x}',t',\wp(u_j^{(n)},p_b^{(n)},p_a^{(n)},c^{(n)},T_j^{(n)}))\right\}+$$
$$+\int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_{jk}(\vec{x}-\vec{x}',t-t')\left\{q_j^a(\vec{x}',t')\Big|_{\partial\Gamma_1}+\wp_j^s(u_j,p_a,p_b,T,c)\Big|_{\partial\Gamma_1}\right\}\right\}\right\}$$
$$c^{(n+1)}(\vec{x},t) \cong K_c^0(t,c_0)+$$
$$+\int_0^t dt'\left\{\iiint_{V_k} d^3x' g_c(\vec{x}-\vec{x}',t-t')\Xi_c(\vec{x}',t',\wp(u_j^{(n)},c^{(n)}))\right\}+$$
$$+\int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_c(\vec{x}-\vec{x}',t-t')\wp_c^s(u_j^{(n)},T^{(n)})\Big|_{\partial\Gamma_1}\right\}\right\}$$
$$p_a^{(n+1)}(\vec{x},t) \cong K_a^0(t,p_a^0)+$$
$$+\int_0^t dt'\left\{\iiint_{V_k} d^3x' g_a(\vec{x}-\vec{x}',t-t')\Xi_a(\vec{x}',t',\wp(u_j^{(n)},p_a^{(n)},p_b^{(n)},c^{(n)},T^{(n)}))\right\}+$$
$$+\int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_a(\vec{x}-\vec{x}',t-t')\wp_a^s(u_j^{(n)},p_a^{(n)},p_b^{(n)},T^{(n)})\Big|_{\partial\Gamma_1}\right\}\right\}$$
$$p_b^{(n+1)}(\vec{x},t) \cong K_b^0(t,p_b^0)+$$
$$+\int_0^t dt'\left\{\iiint_{V_k} d^3x' g_b(\vec{x}-\vec{x}',t-t')\Xi_b(\vec{x}',t',\wp(u_j^{(n)},p_a^{(n)},p_b^{(n)},c^{(n)},T^{(n)}))\right\}+$$
$$+\int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_b(\vec{x}-\vec{x}',t-t')\wp_b^s(u_j^{(n)},p_a^{(n)},p_b^{(n)},T^{(n)})\Big|_{\partial\Gamma_1}\right\}\right\}$$
$$T^{(n+1)}(\vec{x},t) \cong K_j^0(T_0,q_0)+$$
$$+\int_0^t dt'\left\{\iiint_{V_k} d^3x' g_T(\vec{x}-\vec{x}',t-t')\Xi_T^{(n)}(\vec{x}',t',\wp(u_j^{(n)},p_a^{(n)},p_b^{(n)},T^{(n)}))\right\}+$$
$$+\int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_T(\vec{x}-\vec{x}',t-t')\wp_T^s(u_j^{(n)},p_a^{(n)},p_b^{(n)},T^{(n)})\Big|_{\partial\Gamma_1}\right\}\right\}$$

3. FORMULACIÓN DEL PROBLEMA DE CONDICIONES INICIALES Y DE BORDE PARA LA TERMO-ELASTODINÁMICA CON MICRO-TEMPERATURAS EN LA FORMULACIÓN DE SEGUNDO GRADIENTE

Siguiendo a Ieşan (2007) se tienen las siguientes ecuaciones de movimiento

Distribución de campos de desplazamiento en la formulación e segundo gradiente

$$\begin{aligned}
&\rho\partial_t^2 u_j(\vec{x},t) - \rho l_1^2(\partial_t^2 u_j(\vec{x},t))_{,kk} - \mu(1-l_1^2\hat{\nabla}^2)\hat{\nabla}^2 u_j(\vec{x},t) - \\
&\quad -(\mu+\lambda)\hat{\nabla}((1-l_1^2\hat{\nabla}^2)(\hat{\nabla}u_k(\vec{x},t))) + \beta\boldsymbol{M}^E_{jk}(1-l_1^2\hat{\nabla}^2)(1+\partial_t)T_k(\vec{x},t) = \\
&\qquad = -\boldsymbol{S}^{Es}_{jklm}\boldsymbol{\varepsilon}^*_{lm,k}(\vec{x},t) \quad \text{en } R_k \\
&u_j(\vec{x},0) = u_j^0 \;/\; u_j^0 \in (\boldsymbol{H}^1_0(D_k))^3 \;;\; \partial_t u_j(\vec{x},0) = h_j^0 \;/\; h_j^0 \in (\boldsymbol{L}^2(D_k))^3
\end{aligned}$$

Distribuciones de temperaturas según Green-Naghdi tipo II, de micro-temperaturas respectivamente

$$\begin{aligned}
&\rho\partial_t w_j(\vec{x},t) - k_1\hat{\nabla}^2 w_j(\vec{x},t) + k_2 w_j(\vec{x},t) + \beta_1\boldsymbol{M}^E_{jk}(1+\partial_t)T_k(\vec{x},t) = 0 \quad \text{en } R_k \\
&w_j(\vec{x},0) = w_j^0 \;/\; w_j^0 \in (\boldsymbol{H}^1_0(D_k))^3 \\
&\rho c_v\tau\partial_t^2 T(\vec{x},t) - (\boldsymbol{K}^E_{jk}T_{,j}(\vec{x},t))_{,k} + \rho c_v\tau\partial_t T(\vec{x},t) + \\
&\qquad + T_0\bar{\beta}\boldsymbol{M}^E_{jk}(1-l_1^2\hat{\nabla}^2)(\dot{u}_{j,k}(x,t) + \tau\ddot{u}_{j,k}(\vec{x},t)) - \xi_1 w_{j,j}(\vec{x},t) = 0 \quad \text{en } R_k \\
&T(\vec{x},0) = T_0 \;/\; T_0 \in H^1_0(D_k) \;;\; \partial_t T(\vec{x},0) = q_0 \;/\; q_0 \in L^2(D_k)
\end{aligned}$$

Condiciones de borde mixtas

$$\begin{aligned}
&(-\mu\hat{n}_k u_{j,k}(\vec{x},t) + l_1^2\hat{n}_k\hat{\nabla}^2 u_{j,k}(\vec{x},t))\Big|_{\partial\Gamma_1} - \\
&-(\mu+\lambda)\hat{n}_j((1-l_1^2\hat{\nabla}^2)(u_{k,k}(\vec{x},t)))\Big|_{\partial\Gamma_1} + \beta\boldsymbol{M}^E_{jk}(1-l_1^2\hat{\nabla}^2)(1+\partial_t)T(\vec{x},t)\hat{n}_k\Big|_{\partial\Gamma_1} = \\
&\qquad = q_j^a\Big|_{\partial\Gamma_1} / \; q_j^a \in (\boldsymbol{L}^2(\partial\Gamma_1)) \\
&(-k_1 w_{j,k}(\vec{x},t) + \beta_1\boldsymbol{M}^E_{jk}(1+\partial_t)T(\vec{x},t))\hat{n}_k\Big|_{\partial\Gamma_1} = 0 \\
&(-\boldsymbol{K}^E_{jk}T_{,j}(\vec{x},t) + T_0\bar{\beta}\boldsymbol{M}^E_{jk}(1-l_1^2\hat{\nabla}^2)(\dot{u}_j(x,t) + \tau\ddot{u}_j(\vec{x},t)))\hat{n}_k\Big|_{\partial\Gamma_1} - \xi_1 w_j\hat{n}_j\Big|_{\partial\Gamma_1} = 0
\end{aligned}$$

Procediendo como en el modelo anterior, es decir determinadas las funciones de Green, utilizando convenientemente el segundo y tercer teorema de representación de Green-Lagrange, se obtendrán las expresiones integrales de las soluciones en la forma siguiente:

$$\begin{aligned}
u_j(\vec{x},t) = &\iiint_{V_k} d^3x'\{g_{jk}(\vec{x}-\vec{x}',t)\partial_t u_k(\vec{x}',0) - u_k(\vec{x}',0)\partial_t g_{jk}(\vec{x}-\vec{x}',t)\} + \\
&+ \int_0^t dt'\left\{\iiint_{V_k} d^3x' g_{jk}(\vec{x}-\vec{x}',t-t')\Xi_k(\vec{x}',t',\wp(u_j,T))\right\} + \\
&+ \int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\left\{g_{jk}(\vec{x}-\vec{x}',t-t')\left\{q_j^a(\vec{x}',t')\Big|_{\partial\Gamma_1} + \wp_j^s(u_j,T)\Big|_{\partial\Gamma_1}\right\}\right\}\right\} \\
w_j(\vec{x},t) = &\iiint_{V_k} d^3x'\{\bar{g}_{jk}(\vec{x}-\vec{x}',t)w_k(\vec{x}',0)\} + \\
&+ \int_0^t dt'\left\{\iiint_{V_k} d^3x'\bar{g}_{jk}(\vec{x}-\vec{x}',t-t')\bar{\Xi}_k(\vec{x}',t',\wp_w(w_j,T))\right\} + \\
&\qquad + \int_0^t dt'\left\{\oiint_{\partial\Gamma_1} dS\bar{g}_{jk}(\vec{x}-\vec{x}',t-t')\wp_j^s(w_j,T)\Big|_{\partial\Gamma_1}\right\}
\end{aligned}$$

$$T(\vec{x},t) = \iiint_{V_k} d^3x' \{g_T(\vec{x}-\vec{x}',t)\partial_t T(\vec{x}',0) - T(\vec{x}',0)\partial_t g_T(\vec{x}-\vec{x}',t)\} +$$
$$+ \int_0^t dt' \left\{ \iiint_{V_k} d^3x' g_T(\vec{x}-\vec{x}',t-t')\Xi_T(\vec{x}',t',\wp_T(w_j,T)) \right\} +$$
$$+ \int_0^t dt' \left\{ \oiint_{\partial\Gamma_1} dS \left\{ g_T(\vec{x}-\vec{x}',t-t')\wp_T^s(u_j,w_j,T)\Big|_{\partial\Gamma_1} \right\} \right\}$$

Los aproximantes de Picard son:

$$u_j^{(n+1)}(\vec{x},t) = K_j^0(u_j^0,h_j^0) +$$
$$+ \int_0^t dt' \left\{ \iiint_{V_k} d^3x' g_{jk}(\vec{x}-\vec{x}',t-t')\Xi_k(\vec{x}',t',\wp(u_j^{(n)},T^{(n)})) \right\} +$$
$$+ \int_0^t dt' \left\{ \oiint_{\partial\Gamma_1} dS \left\{ g_{jk}(\vec{x}-\vec{x}',t-t') \left\{ q_j^a(\vec{x}',t')\Big|_{\partial\Gamma_1} + \wp_j^s(u_j^{(n)},T^{(n)})\Big|_{\partial\Gamma_1} \right\} \right\} \right\}$$
$$w_j^{(n+1)}(\vec{x},t) = K_j^w(u_j^0,h_j^0) +$$
$$+ \int_0^t dt' \left\{ \iiint_{V_k} d^3x' \bar{g}_{jk}(\vec{x}-\vec{x}',t-t')\bar{\Xi}_k(\vec{x}',t',\wp_w(w_j^{(n)},T^{(n)})) \right\} +$$
$$+ \int_0^t dt' \left\{ \oiint_{\partial\Gamma_1} dS\, \bar{g}_{jk}(\vec{x}-\vec{x}',t-t')\wp_j^s(w_j^{(n)},T^{(n)})\Big|_{\partial\Gamma_1} \right\}$$
$$T^{(n+1)}(\vec{x},t) = K_j^2(T_0,q_0) +$$
$$+ \int_0^t dt' \left\{ \iiint_{V_k} d^3x' g_T(\vec{x}-\vec{x}',t-t')\Xi_T(\vec{x}',t',\wp_T(w_j^{(n)},T^{(n)})) \right\} +$$
$$+ \int_0^t dt' \left\{ \oiint_{\partial\Gamma_1} dS \left\{ g_T(\vec{x}-\vec{x}',t-t')\wp_T^s(u_j^{(n)},w_j^{(n)},T^{(n)})\Big|_{\partial\Gamma_1} \right\} \right\}$$

Los operadores definidos genéricamente como $\wp(u_j^{(n)},T^{(n)}), \wp_j^s(u_j^{(n)},T^{(n)}) \ldots$ etc. deben interpretarse en el sentido que son los términos acoplados que han sido trasladados al miembro de la derecha de cada ecuación y que actúan como forzados externos, todos ellos son acotados, compactos y están definidos en la clausura de los espacios utilizados habitualmente.

En general puede probarse que, para todas las variables descriptoras utilizadas en ambos modelos, dado que $\{u_j\}_{n=1}^{\infty}$ es de Cauchy, entonces

$$\lim_{n\to\infty} \left\{ \sup_{u\in R_k} \left\| \sum_n^{\infty} q_j \{u_j\}_{n=1}^{\infty} - u_j \right\| \right\} = 0 \Longrightarrow u_j^{(n)} \to u \text{ uniformemente } ;\ q_j \in R_0^+$$

4. CONCLUSIONES

En el presente trabajo se analizaron y resolvieron semi-analíticamente, desde la perspectiva de los sistemas de Onsager, un modelo quimi-poroelastico desde la perspectiva de E. Aifantis, y otro de tipo termo-elastodinámico con micro-temperaturas desde la perspectiva de D. Iesan, y R. Quintanilla, en ambos casos se construyen las soluciones semi-analíticas y los aproximantes de Picard, previamente en el primer caso, se definen las funciones de Green asociadas a cada operador, y luego utilizando los teoremas de Green Lagrange, se construyen las ya mencionadas representaciones integrales de las soluciones, el sistema de ecuaciones integrales puede resolverse por múltiples, vías siendo una de las exploradas por este grupo la de los wavelets.

Base de Haar y formula de cuadratura asociada

$$\{t \to \psi_{n,k}(t) = \psi(2^n t - k)\,;\; n \in \mathbb{N}, 0 \leq k < 2^n\}$$

$$\int_{-\infty}^{\infty} dt\, 2^m \psi(2^m t - n)\psi(2^{m_1} t - n_1) = \delta_{m,m_1}\delta_{n,n_1}$$

$$S_N(f) = \frac{(b-a)(d-c)(h-e)}{8N^3} \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} f\{a + (\delta_x/2)(2i-1), c + (\delta_y/2)(2j+1), e + (\delta_z/2)(2k+1)\}$$

Finalmente puede probarse que el error cometido tiene la estructura siguiente

$$d\{\|u_j^* - u_j^{(m)}\|\} \leq \frac{\sigma^{m+1}}{1-\sigma} M_0 \quad ; \quad M_0 = \sup_{\vec{x},t \in R_k} \{\|\boldsymbol{F}(\vec{x}, t, u_j^*)\|\}$$

Las representaciones integrales se expresarían en la forma siguiente

$$\begin{aligned}
\bar{u}_j^{(m)}(\vec{x}, t) = \hat{\bar{u}}_j^0(\vec{x}, t) + \\
+ \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} \Bigg\{ \int_0^t dt' H_{jk}\{x, y, z, t-t', a + (\delta_x/2)(2i-1), c + (\delta_y/2)(2j+1), \\
e + (\delta_z/2)(2k+1)\}\, \psi_k\, \{u_j^{(m+1)}\{a + (\delta_x/2)(2i-1), c + (\delta_y/2)(2j+1), \\
e + (\delta_z/2)(2k+1)\}\}\Bigg\} + \\
+ \delta_x\delta_y\delta_z \sum_{k=1}^{2N}\sum_{j=1}^{2N}\sum_{i=1}^{2N} \Bigg\{ \int_0^t dt' \bar{H}_{jk}\{x, y, 0, t-t', a + (\delta_x/2)(2i-1), c + (\delta_y/2)(2j+1), \\
e + (\delta_z/2)(2k+1)\}\, \bar{\psi}_k\, \{u_j^{(m+1)}\{a + (\delta_x/2)(2i-1), c + (\delta_y/2)(2j+1), \\
e + (\delta_z/2)(2k+1)\}\}\Bigg\}
\end{aligned}$$

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