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PENALIZATION METHODS IN THE NUMERICAL SOLUTION OF THE EIKONAL EQUATION.

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Abstract. *We deal here with a Hamilton-Jacobi-Bellman (HJB) equation with infinite solutions. This multiplicity gives rise to a couple of closely related tasks: the identification of an special solution among all the solutions, and the use of unconventional techniques to obtain approximated solutions which converge to the chosen solution. In this paper we present some results concerning the numerical solution.*

1 INTRODUCTION

Some optimal control problems leads to Hamilton-Jacobi-Bellman (HJB) equations with infinitely many solutions, where only one of these solutions is the optimal cost of the optimal control problem. This fact requires not only the identification of the optimal cost among the solutions but also the use of non-classical techniques to look for approximated solutions which converge to the solution of the original problem. When the HJB equation has a unique viscosity solution, the results due to Barles and Souganidis[1] assure that any discretization scheme which satisfies some suitable properties (monotony, consistence and stability) produces a sequence of "approximate solutions" which converges to the solution of the original problem. Due to the non-uniqueness phenomenon, here it is not possible to use directly the Barles and Souganidis techniques. The analysis of these difficulties was started by Camilli and Grüne.[2] We continue here with the numerical solution of these problems, restricting the analysis to the optimal control problem associated to the eikonal equation

$$\|\nabla u\| = f, \tag{1}$$

when f vanishes at some points. We have presented in Di Marco and González[3] some discretization schemes which improve those presented in Camilli and Grüne.[2] In this work we present a totally discrete procedure to compute the solution of (1). Our scheme of approximation not only brings a sequence of convergent approximations but, in addition, the solution of each totally discrete problem can be computed using iterative algorithms which converge in a finite number of steps

The paper is organized as follows.

In §2, we present the original problem. In §3, we describe some approximations in the continuum realm and its relation with the original solution. In §4, we present two aspects of the discrete time approximation. In §5, we analyze the fully discrete problem and its solution. Finally, in §6, we show two examples of application.

2 DESCRIPTION OF THE PROBLEM AND PRELIMINARY RESULTS

We will follow the presentation of the problem given in Di Marco and González.[3]
We consider a control problem with controlled dynamics

$$\begin{cases} \xi'(t) = q(t), \quad t \geq 0 \\ \xi(0) = x, \end{cases} \tag{2}$$

where $x \in \Omega,\ \Omega$ bounded. We define the exit time τ of the trajectory with initial condition x and velocity $q(\cdot)$

$$\tau = \tau(q(\cdot)) = \inf\{t > 0 : \xi_{q(\cdot)}(t) \notin \Omega\} \tag{3}$$

and we restrict the control policies in the following way: $q(\cdot) \in Q_x$, where

$$Q_x := \{q(\cdot) : (0, \infty) \mapsto \mathbb{R}^N,\ \text{measurable with } \|q(t)\| \leq 1 \text{ in } a.e.\ t,\ \tau < \infty\}.$$

The performance of the control policy $q(\cdot)$ is given by the functional

$$J(x,q) = \int_0^\tau f(\xi(t))\,|q(t)|\,dt + g(\xi(\tau)) \tag{4}$$

which is to say that, for the problems considered here, the instantaneous cost is a function of the current state of the system and it is proportional to the absolute value of the velocity. The optimal cost is

$$U(x) = \inf_{Q_x} J(x,q)\,. \tag{5}$$

2.1 Hypotheses

Throughout the paper, we assume the following hypotheses hold:

Ω is bounded; $f(\cdot) \in Lips(\overline{\Omega})$; $g(\cdot) \in Lips(\overline{\Omega})$; $f(.) : \Omega \longmapsto \mathbb{R}$ is a non negative function.

2.2 HJB equation

It can be proved that $U \in Lips(\overline{\Omega})$ and that U is a solution of the HJB equation associated to this problem. The HJB equation and the boundary condition are:

$$\begin{aligned} \|\nabla U(x)\| - f(x) &= 0, \quad a.e.\ x \in \Omega \\ U(x) &= g(x), \quad x \notin \Omega. \end{aligned} \tag{6}$$

Remark 1 *If $\{x \in \Omega : f(x) = 0\}$ is not empty, there may exist many solutions for this equation. Camilli and Grüne*[2] *proved that U is the maximal solution of (6) in the viscosity sense.*

3 APPROXIMATE CONTINUOUS PROBLEMS

3.1 Approximations with finite horizon

In order to develop some numerical approximation techniques, it is necessary to know whether the infinite horizon problem can be approximated by a family of finite horizon problems. Let $T_0 = \max_{x\in\Omega} d(x, \partial\Omega)$, then $\forall T > T_0$ and $\forall x \in \Omega$, it is possible to define the non-empty set

$$Q_x^T = \{q(\cdot) \in Q_x : \tau(q(\cdot)) \leq T\}. \tag{7}$$

Let us define

$$U^T(x) = \inf_{Q_x^T} J(x,q)\,. \tag{8}$$

The following properties hold:

Proposition 1

1. $U(x) \leq U^{T'}(x) \leq U^{T}(x) \leq U^{T_0}(x),\ \forall x \in \overline{\Omega},\ \forall T' \geq T \geq T_0$
2. $\lim_{T\to\infty} U^{T}(x) = U(x),\ \forall x \in \overline{\Omega}.$

3.2 Approximations by penalizations

In order to obtain convergent approximations, we must deal with penalizations of the original problem.

Definition 1 *Let be $\varepsilon > 0$. We define the penalized functional J_ε and the optimal costs U_ε and U_ε^T*

$$J_\varepsilon(x,q) = \int_0^\tau |q(t)|\ \max(\varepsilon,\ f(\xi(t)))\, dt + g(\xi(\tau)) \tag{9}$$

$$U_\varepsilon(x) = \inf_{Q_x} J_\varepsilon(x,q), \qquad U_\varepsilon^T(x) = \inf_{Q_x^T} J_\varepsilon(x,q). \tag{10}$$

The control problem associated to (9) has strictly positive instantaneous cost. In consequence, the corresponding HJB equation has a unique solution, which is the optimal cost given by (10). This penalization has been proposed by Camilli and Grüne[2] . Some characteristics of the convergence of functions U_ε to the function U is given by the following proposition.

Proposition 2

1. $U(x) \leq U_\varepsilon(x);\ U^T(x) \leq U_\varepsilon^T(x),\ \ \forall x \in \overline{\Omega},\ \forall T \geq T_0.$
2. $U_\varepsilon(x) \leq U_\varepsilon^{T'}(x) \leq U_\varepsilon^{T}(x) \leq U_\varepsilon^{T_0}(x),\ \forall x \in \overline{\Omega},\ \forall T' \geq T \geq T_0.$
3. $U_\varepsilon(x) \leq U_{\varepsilon'}(x),\ \ \forall x \in \overline{\Omega},\ \forall \varepsilon \leq \varepsilon';\ U_\varepsilon^T(x) \leq U_{\varepsilon'}^T(x),\ \ \forall x \in \overline{\Omega},\ \forall T \geq T_0,\ \forall \varepsilon \leq \varepsilon'.$
4. $\lim_{\varepsilon\to 0} U_\varepsilon(x) = U(x),\ \ \forall x \in \overline{\Omega}$. *Besides, $U_\varepsilon(\cdot) \in Lips(\Omega)$, then $U_\varepsilon(\cdot)$ converges uniformly to $U(\cdot)$*

4 TIME DISCRETE PROBLEM

4.1 Control policies discretization

The complete discretization procedure comprises two steps: time discretization and space discretization. We analyze in this section the effect of time discretization.

Definition 2 *For $h > 0$, we define the following elements*

$$Q_{x,h} = \{q(\cdot) \in Q_x : q(\cdot) \text{ constant in each } [\nu h, (\nu+1)h),\ \nu \in \mathbb{N}_0\}, \tag{11}$$

$$U_h(x) = \inf_{Q_{x,h}} J(x,q), \tag{12}$$

$$Q^T_{x,h} = \{q(\cdot) \in Q_{x,h} : \tau(q(\cdot)) \leq T\}, \tag{13}$$

$$U^T_h(x) = \inf_{Q^T_{x,h}} J(x,q). \tag{14}$$

For the functions U_h and U^T_h , the following relations (proved in Di Marco and González[3]) hold:

Proposition 3

1. $U(x) \leq U_h(x),\ \forall x;\ U_h(x) \leq U^T_h(x),\ \forall x,\ \forall T \geq T_0.$
2. $\forall p \in \mathbb{N}, U_{\frac{h}{p}}(x) \leq U_h(x),\ \forall x;\ \forall p \in \mathbb{N}, U^T_{\frac{h}{p}}(x) \leq U^T_h(x),\ \forall x,\ \forall T \geq T_0.$
3. $\lim_{p\to\infty} U^{T+\frac{h}{p}}_{\frac{h}{p}}(x) = U^T(x),\ \forall x;\ \lim_{p\to\infty} U_{\frac{h}{p}}(x) = U(x),\ \forall x.$
4. *The function U_h verifies the following dynamical programming principle: $\forall x \in \Omega$*

$$U_h(x) = \min_{q\in B_1(x)} \left(\int_0^{h\wedge\tau(q)} f(x+qs)\,|q|\,ds + U_h(x + (h \wedge \tau(q))\,q) \right) \tag{15}$$

with boundary condition $U_h(x) = g(x)$, $\forall x \notin \Omega$. Besides U_h is the maximal solution of (15).

Definition 3 *Let P be the operator*

$$\begin{aligned} Pw(x) &= \min_{q\in B_1(x)} \int_0^{h\wedge\tau(q)} f(x+qs)\,|q|\,ds + w(x + (h\wedge\tau(q))\,q), x \in \Omega \\ Pw(x) &= g(x), x \in \partial\Omega \end{aligned} \tag{16}$$

Proposition 4 *P is a non-decreasing operator and $P^\nu w \to U_h$ when $\nu \to \infty$ and*

$$w(x) \geq \begin{cases} T_0 \max\limits_{q\in B_1(x), x\in\Omega} f(x)\,|q| + \max\limits_{x\in\Omega,} g(x), & x \in \Omega \\ g(x), & x \in \partial\Omega \end{cases} \tag{17}$$

Then, we have a theoretical procedure to compute U_h.

4.2 Functional discretization

The approximation U_h is not directly implementable, because it involves the computations of the integrals appearing in (16). To get practical methods, those integrals must also be discretized. We use the following scheme of discretization:

$$\begin{aligned} J_h(x,q) &= \sum_{\nu=0}^{K-1} hf(\xi(\nu h))\,|q(\nu h)| + (\tau(q)-Kh)f(\xi(Kh))\,|q(Kh)| \\ &\quad + g(\xi(\tau(q))). \end{aligned} \tag{18}$$

In that case, defining $V_h(x) = \inf_{Q_{x,h}} J_h(x,q)$, we may have examples where $\lim_{h\to 0} V_h(x) < U(x)$. To eliminate this pathology, we consider a penalization scheme with the special parametrization $\varepsilon = L_f h$.

Definition 4 *Let $q(\cdot) \in Q_{x,h}$, we define*

$$\begin{aligned} \overline{J}_h(x,q) &= \sum_{\nu=0}^{K-1} h\,(L_f h + f(\xi(\nu h))\,|q(\nu h)|) + (\tau(q)-Kh)f(\xi(Kh))\,|q(Kh)| \\ &\quad + g(\xi(\tau(q))), \end{aligned} \tag{19}$$

where $K \in \mathbb{N}_0$ is the maximum integer such that the image of $[0,Kh)$ through ξ belongs to Ω.

$$\overline{V}_h(x) = \inf_{Q_{x,h}} \overline{J}_h(x,q), \qquad \overline{V}_h^T(x) = \inf_{Q_{x,h}^T} \overline{J}_h(x,q). \tag{20}$$

The following properties hold (the proofs can be found in Di Marco and González[3]).

Proposition 5

1. $\overline{V}_h(x) \geq U(x),\ \forall x;\ \overline{V}_h(x) \leq \overline{V}_h^T(x)$.
2. $\forall p \in \mathbb{N},\ \overline{V}_{\frac{h}{p}}(x) \leq \overline{V}_h(x);\ \forall p \in \mathbb{N},\ \forall T \geq T_0,\ \overline{V}_{\frac{h}{p}}^T(x) \leq \overline{V}_h^T(x)$.
3. $\lim_{p\to\infty} \overline{V}_{\frac{h}{p}}^{T+\frac{h}{p}}(x) = U^T(x),\ \forall x;\ \lim_{p\to\infty} \overline{V}_{\frac{h}{p}}(x) = U(x),\ \forall x$.

The last proposition shows us that when we consider the cost functional (19), the sequence of optimal costs converges to U. Moreover, we have that

Proposition 6 *$\overline{V}_h$ verifies the following dynamical programming principle: $\forall x \in \Omega$*

$$\overline{V}_h(x) = \min_{q\in B_1(x),\,x+hq\in\overline{\Omega}} f(x)\,|q|\,h + L_f h^2 + \overline{V}_h(x+hq) \tag{21}$$

with boundary condition $\overline{V}_h(x) = g(x),\ \forall x \notin \Omega$.

Definition 5 *We define the following operators on $C(\overline{\Omega})$:*

$$P_h\Phi(x) = \min_{q\in B_1(x),\, x+hq\in\overline{\Omega}} f(x)\,|q|\,h + L_f h^2 + \Phi(x+hq), \quad x\in\Omega$$
$$P_h\Phi(x) = g(x), \quad x\in\partial\Omega \tag{22}$$

$$\Pi_h\Psi(x) = \min_{q\in B_1(x),\, x+hq\in\overline{\Omega}} \left\{\varphi(q)\Psi(x+hq) + (1-\varphi(q))\right\}, \quad x\in\Omega,$$
$$\Pi_h\Psi(x) = 1-\exp(-g(x)), \quad x\in\partial\Omega. \tag{23}$$

where $\varphi(q) = \exp\left(-\left(L_f h^2 + f(x)\,|q|\,h\right)\right).$

Definition 6 *Let K denote the Kruzkov transformation of functions of $C(\overline{\Omega})$ and its inverse be K^{-1}*

$$\left|\begin{array}{l} z_h(x) = K\left[v_h\right](x) = 1-\exp(-v_h(x)) \\ \\ v_h(x) = K^{-1}\left[z_h\right](x) = -\ln\left(1-z_h(x)\right) \end{array}\right. \tag{24}$$

Lemma 1 *The operator Π_h defined in (23) is contractive. In addition we have: $\Pi_h\circ K = K\circ\, P_h$ and $\Pi_h\circ K^{-1} = K^{-1}\circ\, P_h$.*

Lemma 2 *Let Z_h be the unique fixed point of operator Π_h. The function $\overline{V}_h$ defined in (21) is the unique fixed point of (22) and so, it verifies $\overline{V}_h(x) = -\ln\left(1-Z_h(x)\right)$.*

Proposition 7 $P_h^\nu\Phi \to \overline{V}_h$ *when $\nu\to\infty$, $\forall\Phi\in C(\overline{\Omega})$.*

The last proposition gives us a theoretical procedure to compute $\overline{V}_h$.

5 FULLY DISCRETE PROBLEM AND ITS SOLUTION

To get a complete discrete procedure, we must introduce a space discretization. We consider a mesh Ω_k of size k. We will suppose that Ω is polyhedral and that $\Omega_k \equiv \overline{\Omega}$. Let S_k be the set of mesh nodes and $N_k = card(S_k)$.
We define now a control problem on Ω_k. We will consider as an admissible controlled path any finite sequence of points $\{x_0,\ x_1,\ldots,\ x_\rho\}$ that verify the restriction

$$\begin{cases} x_\mu\in S_k\cap\Omega \quad \mu = 0,1,...,\rho-1, \\ \\ x_\rho\in S_k\cap\partial\Omega, \\ \\ \|x_\mu - x_{\mu-1}\| \le k^{2/3} \quad \mu = 1,...,\rho. \end{cases} \tag{25}$$

Given the initial position x_0, the cost of a trajectory that ends at x_ρ is:

$$F_k(x_0,\ x_1, \ldots,\ x_\rho) = g(x_\rho) + \sum_{\varsigma=1}^{\rho} \left(L_f\, k^{2/3} + f(x_{\varsigma-1})\right) \|x_\varsigma - x_{\varsigma-1}\|\,. \tag{26}$$

We define $w_k(x_0)$ as the optimal cost when the process starts at the initial position x_0, i.e.

$$w_k(x_0) = \min_{x_1,\,\ldots,\,x_\rho} F_k(x_0,\ x_1, \ldots,\ x_\rho). \tag{27}$$

We define the operator P_k

$$P_k\Phi(x) = \begin{cases} \min\left\{\Phi(y) + \left(L_f\, k^{2/3} + f(x)\right)\|y-x\| : \right. \\ \qquad\qquad \left. y \in S_k,\ \|y-x\| \le k^{2/3}\right\} & x \in S_k \cap \Omega, \\ \\ g(x), & x \in S_k \cap \partial\Omega. \end{cases} \tag{28}$$

The relation between the optimal cost w_k and the operator P_k is given by the following poposition:

Proposition 8 *w_k is the unique solution of the equation*

$$\Phi = P_k\Phi. \tag{29}$$

Proof. It is similar to those related to the operator P_h and the Kruskov transformation.

Remark 2 *The equation $\Phi = P_k\Phi$ is the Bellman dynamical programming equation associated to the optimal control of a deterministic Markov chain.*

Corollary 1 $w_k = \lim\limits_{\mu\to\infty} (P_k)^\mu\, \Phi(x)\ \forall \Phi \in \mathbb{R}^{N_k}$.

Proof. It follows at once when we introduce the Kruskov transformation.

Remark 3 *The previous result gives an iterative procedure to get w_k. In fact, it converges in a finite number of iterations. In the numerical applications we have found that the best starting function Φ is $(+\infty)^{N_k}$.*

Proposition 9 $\lim\limits_{k\to 0} w_k(x) = U(x)$.

Proof

Let w_k be the unique solution of the equation (29). So, given $x \in \Omega\,,\ \varepsilon > 0$, there exists a control $q_\varepsilon(\cdot) \in Q_x$ with exit time $T(q_\varepsilon)$ such that

$$J(x, q_\varepsilon(\cdot)) \le U(x) + \varepsilon. \tag{30}$$

Let $\xi_{x,\varepsilon}(\cdot)$ be the trajectory generated by $q_\varepsilon(\cdot)$, then $\xi_{x,\varepsilon}(0) = x$.

We can assume w.l.g. that the chosen control is piecewise constant and then the trajectory is piecewise linear. So, let t_ν be the switching times of $q_\varepsilon(\cdot)$, $\nu = 1,...,\overline{\nu}$.

$$t_0 = 0, \qquad t_{\overline{\nu}+1} = T(q_\varepsilon).$$

We define $\widehat{p} = [2\,T(q_\varepsilon)\,k^{-2/3}]+1$, when $2\,T(q_\varepsilon)\,k^{-2/3}$ is not an integer; and $\widehat{p} = 2\,T(q_\varepsilon)\,k^{-2/3}$ otherwise. Clearly, it holds that $\xi_{x,\varepsilon}(\frac{p}{2}\,k^{2/3}) \in \Omega,\ \forall p = 0, 1, ..., \widehat{p}-1$.

We construct $\xi^k_{x,\varepsilon}(\cdot)$, a trajectory close to $\xi_{x,\varepsilon}(\cdot)$, which joins nodes of Ω_k in the following form:

Let us define

$$\begin{aligned} x_p &= \arg\min\left\{\left\|\xi_{x,\varepsilon}(\frac{p}{2}\,k^{2/3}) - y\right\| : y \in S_k \cap \Omega\right\} \qquad \forall p = 0, 1, ..., \widehat{p}-1. \\ x_{\widehat{p}} &= \arg\min\left\{\|\xi_{x,\varepsilon}(T(q_\varepsilon)) - y\| : y \in S_k \cap \partial\Omega\right\} \end{aligned} \tag{31}$$

It is clear that

$$\left\|\xi_{x,\varepsilon}(\frac{p}{2}\,k^{2/3}) - x_p\right\| \le k \qquad \forall p = 0, 1, ..., \widehat{p}, \tag{32}$$

then, $\forall p = 1, ..., \widehat{p}$

$$\|x_p - x_{p-1}\| \le \left\|\xi_{x,\varepsilon}(\frac{p}{2}\,k^{2/3}) - \xi_{x,\varepsilon}(\frac{p-1}{2}\,k^{2/3})\right\| + 2\,k \le \frac{k^{2/3}}{2} + 2\,k. \tag{33}$$

In the same way we have

$$\|x_p - x_{p-1}\| \ge \left\|\xi_{x,\varepsilon}(\frac{p}{2}\,k^{2/3}) - \xi_{x,\varepsilon}(\frac{p-1}{2}\,k^{2/3})\right\| - 2\,k \ge \frac{k^{2/3}}{2} - 2\,k. \tag{34}$$

For any interval $(\frac{p-1}{2}\,k^{2/3},\ \frac{p}{2}\,k^{2/3})$ where there is not any t_ν, we have that the cost associated to the continuous trajectory is

$$\int_{\frac{p-1}{2}\,k^{2/3}}^{\frac{p}{2}\,k^{2/3}} f(\xi_{x,\varepsilon}(t))dt$$

and the term associated to the discrete trajectory is

$$\left(L_f\,k^{2/3} + f(x_{p-1})\right)\|x_p - x_{p-1}\|\,.$$

So, the difference between homologous terms is

$$\begin{aligned}
&\left| \int_{\frac{p-1}{2}\,k^{2/3}}^{\frac{p}{2}\,k^{2/3}} f(\xi_{x,\varepsilon}(t))dt - \left(L_f\,k^{2/3} + f(x_{p-1})\right) \|x_p - x_{p-1}\| \right| \\
&\qquad \leq \int\limits_{\frac{p-1}{2}\,k^{2/3}}^{\frac{p}{2}\,k^{2/3}} |f(\xi_{x,\varepsilon}(t)) - f(x_{p-1})|\,dt \\
&\qquad\qquad + \left| f(x_{p-1})\frac{k^{2/3}}{2} - \left(L_f\,k^{2/3} + f(x_{p-1})\right) \|x_p - x_{p-1}\| \right| \\
&\qquad \leq \int\limits_{\frac{p-1}{2}\,k^{2/3}}^{\frac{p}{2}\,k^{2/3}} L_f\,\|\xi_{x,\varepsilon}(t) - x_{p-1}\|\,dt \\
&\qquad\qquad + f(x_{p-1}) \left| \frac{k^{2/3}}{2} - \|x_p - x_{p-1}\| \right| + L_f\,k^{2/3}\,\|x_p - x_{p-1}\| \\
&\qquad \leq \int\limits_{\frac{p-1}{2}\,k^{2/3}}^{\frac{p}{2}\,k^{2/3}} L_f\,|k+t|\,dt + M_f\,2k + L_f\,k^{2/3}\left(\frac{k^{2/3}}{2} + 2k\right) \\
&\qquad \leq L_f\left(k + \frac{k^{2/3}}{4}\right)\frac{k^{2/3}}{2} + M_f 2k + L_f\left(\frac{k^{2/3}}{2} + 2k\right)k^{2/3}.
\end{aligned}$$

Then, for k small enough

$$\left| \int\limits_{\frac{p-1}{2}\,k^{2/3}}^{\frac{p}{2}\,k^{2/3}} f(\xi_{x,\varepsilon}(t))dt - \left(L_f\,k^{2/3} + f(x_{p-1})\right) \|x_p - x_{p-1}\| \right| \leq 3M_f\,k.$$

For any interval $\left(\frac{p-1}{2}\,k^{2/3},\,\frac{p}{2}\,k^{2/3}\right)$ which contains one or more switching points, the difference between homologous terms is bounded by $2M_f\,k^{2/3}$. The same bound holds for the final interval $\left(\frac{\widehat{p}-1}{2}\,k^{2/3},\,T(q_\varepsilon)\right)$.

Taking into account the inequality

$$\|\xi_{x,\varepsilon}(T(q_\varepsilon)) - x_{\widehat{p}}\| \leq k,$$

we have that the difference between the final costs is bounded by $L_g k$.

So, finally we get

$$|J(x, q_\varepsilon(\cdot)) - F(x_0,\ x_1, \ldots,\ x_{\widehat{p}})| \leq L_g k + 2M_f\ (\overline{\nu}+1)\,k^{2/3} + 6T\,(q_\varepsilon)M_f\,k^{1/3}.$$

From here we obtain

$$w_k(x_0) \le U(x) + \varepsilon + L_g k + 2M_f\ (\overline{\nu}+1)\, k^{2/3} + 6T(q_\varepsilon) M_f\, k^{1/3}.$$

By computing the limit when k goes to zero and taking in mind the arbitrariness of ε, we obtain the following result

$$\lim_{k\to 0}\ w_k(x_0) \le U(x).$$

On the other hand, let x a state and x_0 like in (25). It is clear that if q is a piecewise constant strategy and x_j the switching points of the trajectory,

$$J(x, q(\cdot)) \le F(x_0,\ x_1, \ldots,\ x_{\widehat{p}}) + M_f k.$$

Then, $U(x) \le F(x_0,\ x_1, \ldots,\ x_{\widehat{p}}) + M_f k$ and so,.

$$U(x) \le\ w_k(x_0) + M_f k.$$

Since (27), we have that

$$U(x) \le \lim_{k\to 0} w_k(x_0).$$

□

6 EXAMPLES

6.1 Example 1

Let $\Omega = [0,1]\times[0,1]$, $f(x_1,x_2) = |x_1 - x_2|$. We take $k = \frac{1}{111}$. We show in the following figure the approximate solution w_k.

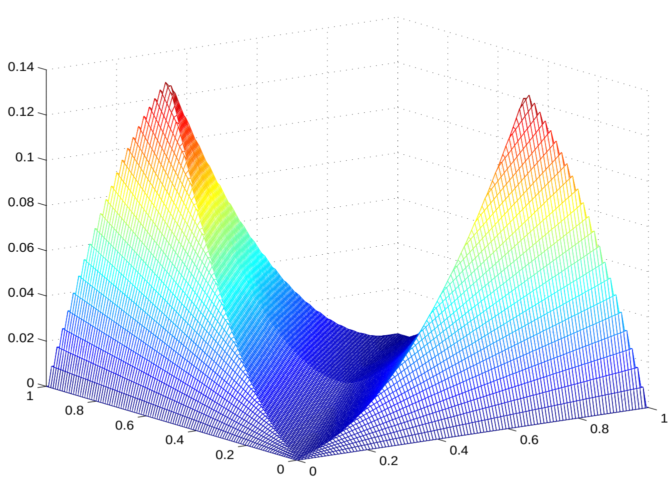

Figure 1: Approximate solution

6.2 Example 2

Let $\Omega = [0,1] \times [0,1]$, $f(x_1, x_2) = \min(|x_1 - .5|, |x_2 - .5|)$. We take $k = \frac{1}{111}$. We show in the following figure the approximate solution w_k.

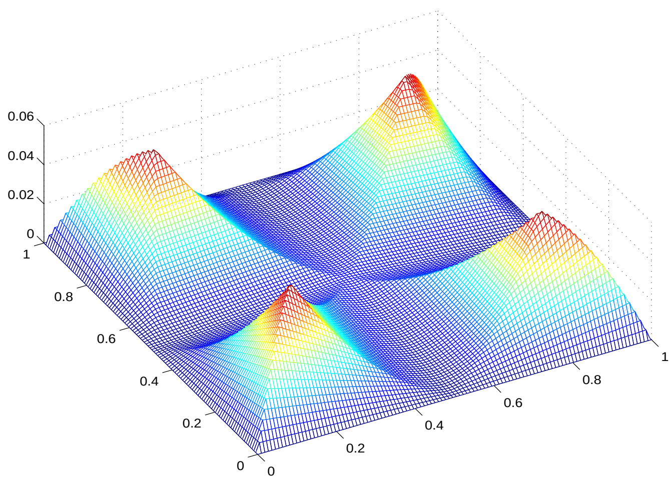

Figure 2: Approximate solution

CONCLUSIONS

We have presented above a complete discrete procedure to approximate the optimal cost of a singular optimal control problem. We have shown that using penalization, finite horizon or discrete time controls we get convergent approximations. This convergence may be lost once the discretization of the dynamic of the system is introduced. In order to recover the convergence property we can use a penalization of the instantaneous cost. This is done in this paper using a penalization of order h, being h the time-step employed. Finally, after having introduced the space discretization (a mesh of size k for the set Ω), we have presented a scheme of approximation that converge in a finite number of steps. In addition, we have proved the convergence of the discrete solutions to the maximum solution of (1). We also have shown some numerical examples.

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