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TOEPLITZ PRECONDITIONER FOR PSEUDO-TOEPLITZ MATRICES.

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abstract *Many problems in Economic and Ingineering involve linear systems with Toeplitz matrices. The Preconditioned Conjugated Gradient (PCG) method for symmetric positive definite Toeplitz systems is highly efficient when the preconditioner is well chosen. In this work we are interested in solving linear system with matrices, which we call pseudo-Toeplitz that present certain similarity with Toeplitz Matrices. These matrices can be seen as perturbated Toeplitz matrices. Problems with this type of matrix can be encountered in different areas especially those expressed in Partial Differencial Equations. Applying the PGC to systems with Pseudo-Toeplitz matrices where the preconditioner corresponds to the unperturbated Toeplitz matrices gives very encouraging results.*

Introduction

Presentation of the problem

Let A be a symmetric definite positive matrix, and b a real vector. The Preconditioned Conjugated Gradient algorithm to solve the linear system

$$Ax = b \tag{1}$$

is given by

1. $x_0 = 0$, $r_0 = b$, $Cp_0 = r_0$, $z_0 = p_0$
2. If $r_k = 0$, $x = x_k$ stop.
   Othewise
   (a) $\alpha_k = \frac{\langle r_k, z_k \rangle}{\langle Ap_k, p_k \rangle}$
   (b) $x_{k+1} = x_k + \alpha_k p_k$
   (c) $r_{k+1} = r_k - \alpha_k Ap_k$
   (d) $Cz_{k+1} = r_{k+1}$
   (e) $\beta_{k+1} = \frac{\langle r_{k+1}, z_{k+1} \rangle}{\langle r_k, z_k \rangle}$
   (f) $p_{k+1} = z_{k+1} + \beta_{k+1} p_k$

In this algorithm the matrix C is the preconditioner. Usually C is chosen in a way that the linear system of step d is easy to solve.

Case of Toeplitz matrices

A square symmetric matrix A of order n is said to be a Toeplitz matrix if there exist $a_i, i = 0, \ldots, n-1$ reals such that

$$A_{ij} = a_{|i-j|}, \qquad i, j = 0, \ldots, n-1 \tag{2}$$

Many precoditioners have been suggested when A is a Toeplitz matrix to solve (1). Often circulant matrices are chosen. This choice is due to the fact that the solution of the system $Cz = r$ can be done in $O(n \log_2 n)$ operations if n is a power of 2. Other non circulant preconditioners proved their efficiency as the one given by E. Boman and I. Koltracht based on the fast sine transform which we will use in this work.

Generally, a good preconditioner is constructed such that

1. The spectrum of $C^{-1}A$ is clustered.
2. C can be computed in $O(n \log_2 n)$ operations
3. Solving $Cr = z$ requires $O(n \log_2 n)$ operations.

Examples of preconditioners for Toeplitz Systems

Raymond and Mikhael gave a very rich list of preconditioners used for Toeplitz systems in their paper [3]. We mention some of them:

Strang's preconditioner

The preconditionner given by Strang [6] is the solution of the problem

$$\min_{C\in\mathcal{C}} \|C-A\|_1,$$

where $\mathcal{C}$ represent the set of circulant matrices. This circulant preconditioner is given by the vector s defined by

$$s_j = \begin{cases} a_j & \text{if} \quad 0 < j \leq [n/2], \\ a_{j-n} & \text{if} \quad [n/2] < j < n, \\ s_{n+j} & \text{if} \quad 0 < -j < n \end{cases}$$

Chan's preconditioner

Another preconditioner given by Chan [4] is the solution of the problem

$$\min_{C\in\mathcal{C}} \|C-A\|_F,$$

where $\|.\|_F$ is Frobenius norm. The vector defining this circulant preconditioner is

$$c_j = \begin{cases} \dfrac{(n-j)a_j + j a_{j-n}}{n} & \text{if} \quad 0 \leq j \leq n, \\ c_{n+j} & \text{if} \quad 0 < -j < n \end{cases}$$

Sine Transform Based Preconditioner(STBP)

As opposed to the examples given above, which are circulant, the Sine Transform Based Preconditioner (STBP), given by E. Boman and I. Koltracht [1] is not. Let us give a detailed description of its construction.

Let S_1 be the sine matrix:

$$S_1 = \sqrt{\frac{2}{n+1}} \left[\sin\left(\frac{ij\pi}{n+1}\right)\right]_{i,j=1}^{n}.$$

and D_{S_1} be the n dimensional vector space of matrices M such that $D_{S_1} M D_{S_1}^{-1}$ is diagonal.

A basis of D_{S_1} is the set of matrices $\{\zeta_p\}_{p=0}^{n-1}$ defined by:

$$\zeta_p(i,j) = \begin{cases} 1 & \text{if} \quad |i-j| = p \\ -1 & \text{if} \quad i+j = p \\ -1 & \text{if} \quad i+j = 2(n+1)-p \\ 0 & \text{otherwise} \end{cases}$$

The suggested preconditioner P_β is :

$$P_\beta = \sum_{i=0}^{\beta} a_i \zeta_i \tag{3}$$

where $a_i, i = 1 \ldots \beta$ are the coefficients of the first row of A, and β is the bandwidth of A. It is shown in that all the conditions of preconditioner are satisfied.

Pseudo-Toeplitz matrices

Definition 0.0.1 *Let A be a symmetric square matrix of order n. A is said to be pseudo Toeplitz of type 1 if there exists n coefficients $c_0, c_1, \ldots, c_{n-1}$ such that for all diagonal $(a_0, a_2, \ldots, a_{k-1})$, $k = 0, ..n-1$*

$$\forall i = 1, \ldots, k \qquad a_i = c_k \text{ or } a_i = 0 \tag{4}$$

A is said to be pseudo-Toeplitz of type 2 if for all diagonal $(a_0, a_2, \ldots, a_{k-1}),\ k = 0, \ldots, n-1$ *there exists a small* $\varepsilon_k > 0$ *such that*

$$\forall i, j = 0, \ldots, k-1 \qquad \|a_i - a_j\| \leq \varepsilon_k \tag{5}$$

In other words, pseudo-Toeplitz matrix of type 1 is a Toeplitz matrix where some entries are replaced by 0. A typical case where these matrices occur is in solutions of Partial Differetial Equations with Finite Difference method. Pseudo-Toeplitz of type 2 is a sum of a Toeplitz matrix T and a matrix Π, called the perturbation matrix, such that

$$\max_{i,j=1,\ldots,n} \Pi_{ij} \leq \max_{k=1,\ldots,n} \varepsilon_k.$$

Note that every matrix can be considered as pseudo-Toeplitz of type 2 taking $\varepsilon_k, k = 0, \ldots, n-1$ big enough. We find this class of matrices in Finite Element methods particularly when regular discretization is used as we see further on.

Preconditioning pseudo-Toeplitz systems

Let A be a pseudo-Toeplitz definite positive matrix. From the previous paragraph we know that

$$A = T + \Pi \tag{6}$$

where T is a Toeplitz matrix and Π is the perturbation matrix. We will solve the system $Ax = b$ with PCG using as preconditioner the one associated to the Toeplitz matrix T. In the PCG algorithm to solve the system $Tx = b$, the only step that changes is the one where the product

$$\langle Tp_k, p_k \rangle \tag{7}$$

is computed. This number is replaced by

$$\langle Tp_k, p_k \rangle + \langle \Pi p_k, p_k \rangle. \tag{8}$$

In the next cases, we use the Sine Transform Based Preconditioner to evaluate the convergence of the PCG for pseudo-Toeplitz systems.

Perturbated Toeplitz Matrices

To test the efficiency of this preconditioning method, we will introduce a perturbation on a given Toeplitz matrix and compare the results to non preconditioned conjugated gradient and to non perturbated PCG. The way we perturbate the Toeplitz matrix is the following. Let δ be a positive real. Randomly, we generate a symmetric matrix Π so that

$$\max_{i,j=1,\ldots,n} |\Pi_{ij}| = \delta.$$

with the risk of losing the positive definite caracter of the system matrix $T + \Pi$. The example studied was the Toeplitz matrix defined by :

$$T = \left[\frac{1}{(|i-j|+1)^{1.1}} \right]_{i,j=1}^{n}$$

The results are summarized in the table (1). Different values of n, dimension of the problem were tested (first column). Due to the random charater of the perturbation Π, for every n and δ given, the program has been run five times (third column). The fourth and fifth column represent the number of iterations without and with the preconditioner respectively. In the last column we present the results of the PGC for the Toeplitz system $Tx = b$.

Application to Finite Difference and Finite Element problems

Let us consider the elliptic problem:
to find u, solution of:

$$(\pi)\begin{cases} -\Delta u + cu = f & \text{in} \quad [0, \quad 1] \times [0, \quad 1] \\ u = g_1 & \text{in} \quad \Gamma_1 \\ \frac{\partial u}{\partial n} = g_2 & \text{in} \quad \Gamma_2 \end{cases}$$

where f, g_1, g_2 are given functions, Γ_1, Γ_2 form the boundary of the square $[0, \quad 1] \times [0, \quad 1]$, c is positve and $\frac{\partial}{\partial n}$ is the normal derivative.

Finite Difference case

It is obvious that solving (π) with the five points scheme of Finite Difference method leads to a pseudo-Toeplitz matrix of type 1. We discretize the edges in both x and y directions using N equidistant interior nodes. We number the nodes from left to right and from the bottom to top. In this case :

$$\Pi_{i,j} = \begin{cases} 1 & \text{if} \quad i+1 \equiv 0[N] \quad \text{and} \quad j = i+1; \\ 1 & \text{if} \quad i-1 \equiv 0[N] \quad \text{and} \quad j = i-1; \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

If x is n-vector, multilplying Π by x gives :

$$(\Pi x)_k = \begin{cases} x_{N(i-1)} & \text{if} \quad k = N(i-1)+1, i = 1, .., N-1, \\ x_{Ni+1} & \text{if} \quad k = Ni, i = 1, .., N-1 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

and

$$\langle \Pi x, x \rangle = 2 \sum_{i=1}^{N-1} x_{iN} x_{iN+1}. \tag{11}$$

Hence the number of operations of (11) is enormously reduced since it requires only $2N-2$ while the size of the matrix is N^2. We used STBP for this problem. We also solved the problem using the SSOR preconditioner with parameter $w = 1$. The results obtained are summarized in table 2

Note that the smaller the ratio between the bandwidth and the matrix size is, the better the convergence using this preconditioner becomes.

Finite Element case

We solve the problem π with the Finite Element method using the P1 triangle. This kind of preconditioning is particularly useful when the triangularization is regular as given in figure (1)

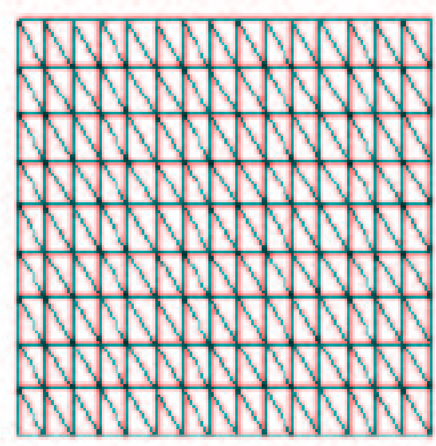

Figure 1: Regular Finite Element Discretization

By regular we mean that the triangles have the same area and can be seen as horizontal or vertical translation of two triangles. With this type of discretization and convenient numbering of the nodes the coefficients of the stiffness matrix often appear repeatedly along its diagonals. This property makes the resulting matrix an excellent example of pseudo-Toeplitz matrices.

To calculate the preconditioner associated to the Toeplitz matrix we proceed by extracting from the finite element matrix the Toeplitz matrix as follows: for every diagonal i of the matrix we choose the coefficient a_i that appears more frequently in this diagonal $i = 0, \ldots, n-1$ and then the Toeplitz matrix will be $a_{|i-j|}, i, j = 0, \ldots, n-1$.

We compared this preconditioner to the SSOR one with $\omega = 1$. We used the Finite Element Library MODULEF to solve the problem. The results obtained are summarized in the table (3)

We see that this preconditioner is particuarly efficient for high dimension problems.

Conclusion

In this work many numerical experiments have been carried out. The idea of using the pseudo-Toeplitz character of PDE problems is very encouraging given that some results are much better than classical preconditioners like SSOR. Our goal in the future is to apply these experiments to other problems such elasticity and to use the block decomposition preconditioning of the resulting linear systems.

n	δ	try	CG	PCG	non perturbated PCG
100	0.1	1	63	34	5
100	0.1	2	61	33	5
100	0.1	3	59	34	5
100	0.1	4	62	35	5
100	0.1	5	64	37	5
100	0.01	1	21	7	5
100	0.01	2	21	7	5
100	0.01	3	22	7	5
100	0.01	4	22	7	5
100	0.01	5	22	7	5
500	0.01	1	31	9	5
500	0.01	2	31	9	5
500	0.01	3	30	8	5
500	0.01	4	31	9	5
500	0.01	5	31	9	5

Table 1: Results of Perturbated matrices using STBP

dimension n	Bandwidth	PCG	SSOR $\omega = 1$
500	250	31	11
500	100	23	26
500	50	18	48
500	25	16	78
500	10	19	146
1000	500	50	11
1000	250	34	20
1000	100	24	55
1000	50	19	91
1000	25	19	149
1000	10	28	287
2000	1000	87	11
2000	500	54	21
2000	250	36	43
2000	100	24	111

Table 2: Results of the Finite Difference problem solved as a pseudo-Toeplitz problem using STBP compared with the SSOR preconditioner.

problem size n	GC	PCG (with STBP)	SSOR $\omega = 1$
150	56	25	22
200	92	32	25
225	75	28	26
250	115	35	35
288	83	30	28
320	86	30	29
350	118	35	35
375	118	35	36
400	101	32	32
475	121	35	36
500	121	35	37
525	163	41	47
572	128	36	39
750	147	39	46
1250	235	49	67

Table 3: Results of PCG with (STBP) and SSOR for the Finite Element method.

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