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JACKSON'S INEQUALITIES FOR *h-p* CLOUDS AND ERROR ESTIMATES

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Abstract. The interest in meshfree methods for solving boundary-value problems has grown rapidly in recent years. A meshless method that has attracted much interest in the community of computational mechanics is the *h-p* clouds method. For this kind of applications it is fundamental to analyze the orders of approximation. In this this paper we prove a Jackson type inequality for *h-p* cloud functions. This inequality set up a general framework for the theoretical analysis of high order error estimates of the *h-p* clouds method, with the same remarkable features of Finite Element theory.

1 INTRODUCTION

In spite of the great success of the finite element method as effective numerical tool for the solution of boundary-value problems in complex domains, there has been a growing interest in meshless method over the last decade. The automatic generation of 3-D meshes presents significant difficulties in the analysis of engineering systems and the development of techniques which do not require the generation of a mesh is very appealing. In meshless method *h-p* (spectral) types of approximations are built around a collection of nodes sprinkled within the domain on which a boundary-value problem has been posed. Associated with each node, there is an open set that forms the support for the approximation basis functions built around the node. The boundary-value problem is then solved using these *h-p* functions and a Galerkin method.

A meshless method which has very attrative features is the *h-p* clouds of C. A. Duarte and J. T. Oden. The basic idea of the method is to multiply a partition of unity by polynomials or other class of functions. The resulting functions, called *h-p* clouds, have good properties, such as high regularity and compactness; and linear combinations of these functions can represent polynomials of any degree. This property allows the implementation of p and *h-p* adaptivity. For this kind of applications, it is fundamental to analyze the orders of approximation in the context of Sobolev spaces. A partial result in this direction was given in the seminal works of C. A. Duarte and J. T. Oden.[4–6]

The aim of this paper is to give insight into smoe recent results in the theoretical analysis of high order error estimates for the *h-p* clouds method. The results are proven in details in C. Zuppa.[14,15]

A key ingredient for error estimate in Sobolev spaces is polynomial approximation of Sobolev functions. Here, we have used Verfürth's approach since it provides Jackson's type inequalities with better bounds. In convex domains for example, the bounds do not depend on the eccentricity.

The paper is organized as follows. In Section 1 we present a fundamental concept in this work: the m-modified *h-p* approximation operators. Section 2 deals with polynomial approximation of functions in Sobolev spaces in star-shaped domains. In Section 3 we analyze error estimates for *h-p* approximation operators modified with Verfürth's averaged polynomials. Finally, applications to *h-p* cloud functions are discussed; in particular, we derive error estimates for the approximate solutions of boundary-value problems.

We do not discuss here numerical results but it is worthwhile to mention that there is an extensive literature in computational mechanics practitioners confirming the theoretical error estimates (to begin with, see ,[4–7] for example).

2 *H-P* FORMULAS

Let Ω be an open bounded domain in $\mathbb{R}^n$ and Q^N denote an arbitrarily chosen set of N points $x_i \in \overline{\Omega}$ referred to as *nodes*:

$$Q^N = \{x_1, x_2, ..., x_N\}, \quad x_i \in \overline{\Omega}$$

Let $\mathcal{I}_N := \{\omega_i\}_{i=1}^N$ denote a finite open covering of $\overline{\Omega}$ such that $x_i \in \omega_i$, $i = 1, ..., N$, and let $\mathcal{S}_N := \{\mathcal{W}_i\}_{i=1}^N$ be a *partition of unity* subordinate to $\mathcal{I}_N$ That is, $\mathcal{S}_N$ is class of functions with the following properties:

$\mathcal{W}_i \in C_0^s(\mathbb{R}^n), \quad s \geq 0$ or $s = +\infty$

spt $(\mathcal{W}_i) = \overline{\omega}_i$

$\mathcal{W}_i(x) > 0, \quad x \in \omega_i$

$\sum_{i=1}^N \mathcal{W}_i(x) = 1, \quad \forall x \in \overline{\Omega}.$

In particular, for every $x \in \overline{\Omega}$, there is at least one $\mathcal{W}_j$ such that $\mathcal{W}_j(x) > 0$.

Remark 2.1 *Here, we assume $\mathcal{W}_i \geq 0$ for simplicity, but this is not an essential requirement. With obvious modification, the general case could also be treated.*

Assumption: In what follows, we shall not deal with questions related to the differentiability of functions $\mathcal{W}_i$. For the sake of simplicity, from now on, we make the assumption that $\mathcal{W}_i \in C_0^\infty(\mathbb{R}^n)$, $i = 1, ..., N$.

The sets ω_i, $i = 1, ..., N$ are called *clouds* in meshless methods community (see[6] for example).

Notation 2.2 *The diameter of ω_i, $d_i := \sup_{x,y\in\omega_i}\{||x-y||\}$ and*

$$h := \max_{i=1,\dots,N}\{d_i\}$$

will be key ingredients in error estimates. For future use, we set $\widetilde{\omega}_i := \omega_i \cap \Omega$ and $\widehat{i} := \{j : \omega_i \cap \omega_j \neq \emptyset\}$.

Let m be any integer ≥ 0. For $i = 1, ..., N$, $\mathcal{P}_i^m$ denotes the vector space of q-Taylor polynomials at x_i

$$\mathcal{P}_i^m := \left\{ Q : Q(x) = \sum_{0\leq|\nu|\leq m} a_\nu (x - x_i)^\nu \right\}.$$

Definition 2.3 *Let $\mathcal{F}$ be some space of functions. Given a linear operator*

$$\mathcal{T}^m : \mathcal{F} \to \prod_{i=1}^N \mathcal{P}_i^m, \tag{1}$$

the associated m-modified h-p approximation operator is the linear operator $\mathcal{ST}^m : \mathcal{F} \to C^\infty(\overline{\Omega})$ defined by

$$\mathcal{ST}^m(u) := \sum_{i=1}^N T_i^m(u) \cdot \mathcal{W}_i, \quad u \in \mathcal{F} \text{ and } \mathcal{T}^m(u) = (\mathcal{T}_i^m(u))_{i=1,\dots,N}.$$

In this work, we are mainly interested in the case where $(\mathcal{F}, ||\cdot||_{\mathcal{F}})$ is some Sobolev space of functions over Ω. In this context, the problem is to estimate the approximation error:

$$||u - \mathcal{ST}^m(u)||_{\mathcal{F}}\,.$$

In the general case, however, $\mathcal{ST}^m$ is not an approximation operator in the classical sense:

Example 2.4 $\mathcal{F} = \prod_{i=1}^{N} \mathcal{P}_i^m$ *and* $\mathcal{T}^m =$ *identity. In this case,* $\mathcal{ST}^m(\mathcal{F}\,) \subset C^\infty(\overline{\Omega})$ *is the space* $\mathcal{F}^{\,0,m}$ *of h-p clouds functions of Duarte-Oden.*[5,6] *If* $\widetilde{\mathcal{ST}}^m$ *is another m-modified h-p approximation operator and* $u \in (\mathcal{F}, ||\cdot||_{\mathcal{F}})$*, the following inequality is meaningful:*

$$\inf_{U \in \mathcal{F}^{\,0,m}} ||\, u - U\,||_{\mathcal{F}} \leq ||u - \widetilde{\mathcal{ST}}^m(u)||_{\mathcal{F}}\,.$$

These are the kinds of application to h-p clouds functions that we will investigate in the last section.

Section 4 is dedicated to our main example of *m-modified h-p approximation operator*. There, the operator $\mathcal{T}^m$ is built with the help of local averaged Taylor polynomials of Sobolev functions.

3 POLYNOMIAL APPROXIMATION IN SOBOLEV SPACES

Given $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$ we denote, as usual,

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!.$$

For $p \geq 1$ and $m \in \mathbb{N}_0$, we call $W_p^m(\Omega)$ the Sobolev space which consists of all the functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq m$. Given $j \in \mathbb{N}_0$, we define

$$|\,u\,|_{\,j,p} = \left(\sum_{|\alpha|=j} ||D^\alpha u\,||^p_{L^p(\Omega)}\right)^{1/p};$$

therefore, the usual norm in $W_p^m(\Omega)$ is defined by

$$||\,u\,||_{m,p} = \left(\sum_{j=0}^{m} |\,u\,|^p_{j,p}\right)^{1/p}.$$

If $p = 2$, we denote as usual $W_2^m(\Omega) = H^m(\Omega)$. When explicit reference to the domain is needed, we denote $||\,u\,||_{m,p} = ||\,u\,||_{\Omega,m,p}$ and $|\,u\,|_{\,j,p} = |\,u\,|_{\Omega,\,j,p}$.

Let $U \subset \mathbb{R}^n$ be an open set with diameter d_U. In this section, we are interested in sharp upper bounds on the constant $c_{m,j}$ in the Jackson-type inequalities

$$\sup_{u \in W_p^{m+1}(U)} \inf_{p \in \mathcal{P}^m} \frac{|u-p|_{j,p}}{|u|_{m+1,p}} \leq c_{m,j} d_U^{m+1-j} \quad \forall \quad 0 \leq j \leq m, \tag{2}$$

when U is star-shaped w.r.t. at least a point in U. Here, $\mathcal{P}^m$ is the space of all polynomials in n variables of degree at most m. The best estimates of $c_{m,j}$, which are known to us, are due to Verfürth[13] and Durán.[8] Here, we shall follow the Verfürth's program because his bounds do not depend on eccentricity in case U is a convex set. Moreover, for non-convex domains with a re-entrant corner, the bounds are uniform w.r.t. the exterior angle.

Let $B \subset U$ be a set of positive measure $|B|$. For any integer m and $p \geq 1$, there exists a projection operator Q_B^m of $W_p^m(U)$ onto $\mathcal{P}_m$ which has the following properties

$$D^\beta(Q_B^m u) = Q_B^{m-j}(D^\beta u), \tag{3}$$

$$\int_B D^\beta(u - Q_B^m u)(y)\, dy = 0 \tag{4}$$

for all $u \in W_p^m(\Omega)$, all $0 \leq j \leq m$, and all $\beta \in \mathbb{N}^n$ with $|\beta| = j$. We denote by

$$\pi_B(\, f\,) := \frac{1}{|B|} \int_B f(y)\, dy$$

the mean value of f w.r.t. B.

For any $u \in W_p^m(\Omega)$ we recursively define polynomials $q_B^m, ..., q_B^0$ in $\mathcal{P}_m$ by

$$q_B^m(u) := \sum_{|\alpha|=m} \frac{1}{\alpha!} x^\alpha \pi_B(D^\alpha u)$$

and for $k = m, m-1, ..., 1,$

$$q_B^{k-1}(u) := q_B^k(u) + \sum_{|\alpha|=k-1} \frac{1}{\alpha!} x^\alpha \pi_B(D^\alpha u - q_B^k(u)),$$

we set

$$Q_B^m\, u := q_B^0(u).$$

Definition 3.1 *Given $u \in W_p^m(\Omega)$ and $P_{m,B}\, u$, the remainder term is $R^m u := u - P_{m,B}\, u$.*

Definition 3.2 *A set U is star-shaped w.r.t. a set B if, for all $x \in U$, the closed convex hull of $\{x\} \cup B$ is a subset of U.*

The star-shaped condition is a key ingredient in polynomial approximation in Sobolev spaces. In order to state Verfürth's results for non-convex but star-shaped domains, we need to state some more definitions. For $z \in U$, we define

$$\chi(z) := \max_{y \in \partial U} ||y - z|| \,/\, \min_{y \in \partial U} ||y - z|| \,.$$

Now, assume that U is non-convex but star-shaped w.r.t. at least one point and let $\mathcal{S}_U := \{\, z \in U : U$ is star-shaped w.r.t. $z\}$. It is clear that there exists a point $z_m \in U$ such that $\chi(z_m) = \min_{z \in \mathcal{S}}\{\chi(z)\}$. Then, the number $\varkappa$ is defined by

$$\varkappa := \chi(z_m).$$

The main Verfürth's result in[13] is:

Theorem 3.3 *Let U be a domain star-shaped w.r.t. at least one point. For $1 \leq p \leq \infty$ and $m \in \mathbb{N}_0$, there exist constants $c_{m,j}$, $0 \leq j \leq m$, such that*

$$||\, u - Q^m_{B_U}\, u \,||_{j,p} \leq c_{m,j}\, d_U^{m+1-j} \,|\, u \,|_{m+1,p}\,, \quad \forall u \in W_p^{m+1}(U).$$

When U is a convex domain, $B_U = U$ and $c_{m,j} = c_{m,j}(n,m)$, i.e., the bounds $c_{m,j}$ depend only on n and m. In the non-convex case, $B_U = B(z_m, \varrho)$, $\varrho =$dist$(z_m, \partial U)$, and $c_{m,j} = c_{m,j}(n, m, \varkappa)$.

In order to apply this result we need the star shaped condition at least locally:

A0 For every $i = 1, ..., N$, $\widetilde{\omega}_i$ is star-shaped w.r.t. at least a point $y_i \in \widetilde{\omega}_i$.

A trivial condition guarantying **A0** is:

Example 3.4 *Ω and ω_i, $i = 1, ..., N$ are convex sets.*

A less trivial example is:

Example 3.5 *If Ω has a Lipschitz continuous boundary $\partial\Omega$,[1] and every ω_i, $i = 1, ..., N$, is a convex set of diameter d_i, it is a geometrical fact that there exists $\varrho_\Omega > 0$ such that $(\Omega, \mathcal{I}_N)$ satisfies **A0** if $\sup_{i=1,...N}\{d_i\} \leq \varrho_\Omega$.*

A0L Ω has a Lipschitz continuous boundary $\partial\Omega$.

A0 is a key ingredient in our estimates. From now on, and without explicit mention, we assume:

Condition A0 is satisfied.

4 TAYLOR AVERAGED *H-P* FORMULAS

For each i, $i = 1, ...N$, we can choose a subset $B_i \subset \widetilde{\omega}_i$ where Verfürth's projection operator applies (see theorem 3.3). We assume that $B_i = \widetilde{\omega}_i$ in case $\widetilde{\omega}_i$ is a convex set.

Given an integer $m \geq 0$ and $p \geq 1$, we will define now an *m-modified h-p approximation operator* $\mathcal{ST}^m : W_p^{m+1}(\Omega) \to C^\infty(\overline{\Omega})$, using Verfürth's projection operators.

Let $u \in W_p^{m+1}(\Omega)$. For $i = 1, ..., N$, we set $Q_i^m u = Q_{B_i}^m u$. Then,

$$\mathcal{ST}^m(u) = \sum_{i=1}^{N} Q_i^m u \, \mathcal{W}_i \,. \tag{5}$$

Remark 4.1 *Note that, even when $u \in W_p^{m+1}(\Omega) \cap C(\overline{\Omega})$, $\mathcal{ST}^m(u)(x_i) \neq u(x_i)$. Then, $\mathcal{ST}^m(u)$ is not an interpolant of u.*

We are interested in estimating the error $u - \mathcal{ST}^m(u)$ in Sobolev norms. Several constants are cornerstones in obtaining error estimates. We begin with:

A1 A measure of the overlap of *clouds:*

$$M = \sup_{i=1,...,N} \{\# \widehat{i}\} \,,$$

where $\#S$ denotes the number of elements in a finite set S.

Remark 4.2 *Other authors[4, 10, 11] use the pointwise condition **A1P**:*

$$M = \sup_{x \in \overline{\Omega}} \{\#(j : x \in \omega_j)\}.$$

*However, **A1** and **A1P** are different requirements. I could not obtain the results here assuming **A1P**.*

The following result will be useful in passing from local to global results.

Lemma 4.3 *Assume condition **A1** holds and let $f, g \in L^1(\Omega)$ be two positive functions. If*

$$\int_{\omega_i} f(x)dx \leq \sum_{j \in \widehat{i}} \int_{\omega_j} g(x)dx \,, \quad \forall i : i, ..., N$$

then

$$\int_{\Omega} f(x)dx \leq M^2 \int_{\Omega} g(x)dx,$$

*where M is the constant in condition **A1**.*

We define now other crucial constants.

For every i, $i = 1, ...N$ we have a set of bounds $c_{m,j}(i)$ given by theorem 3.3.

A2 For every i, $i = 1, ...N$, we define

$$CM_{m,i} := \max_{0 \le j \le m} \{c_{m,j}(i)\}.$$

We also need:

A3 $C_{D,m,i} > 0$, $i = 1, ..., N$, are constants such that

$$||D^\beta \mathcal{W}_j||_{L^\infty} \le \frac{C_{D,m,i}}{d_j^{|\beta|}}, \qquad |\beta| \le m, \, \forall j : \; j \in \widehat{i} \; .$$

Our first local error estimate is:

Theorem 4.4 *Assume* ***A1*** *and* ***A2****; and let* $p \ge 1$, $l \le m$. *Then,*

$$|\, u - \mathcal{ST}^m(u)\,|_{\widetilde{\omega}_i,l,p} \le C_{m,l,i}\, h_i^{m+1-l} |\, u\,|_{\widehat{\omega}_i,m+1,p}\,, \quad \forall u \in W_p^{m+1}(\Omega),$$

where

$$C_{i,m,l} = M^{\,1/p} C(n,m)\, C_{D,m,i}\, CV_{m,i}\, (\#\{\alpha : |\alpha| = l)^{1/p}\,,$$

$h_i := \max_{j \in \widehat{i}}\{d_j\}$, *and* $CV_{m,i} = \max_{j \in \widehat{i}}\{CM_{m,j}\}$. *In particular, there exists a constant*

$$C_{m,i} = C(C_{m,0,i}, ..., C_{m,l,i}),$$

such that

$$||\, u - \mathcal{ST}^m(u)\,||_{\widetilde{\omega}_i,l,p} \le C_{m,i}\, h_i^{m+1-l} |\, u\,|_{\widehat{\omega}_i,m+1,p}, \quad \forall u \in W_p^{m+1}(\Omega).$$

In deriving uniform error estimates, conditions **A2** and **A3** are changed to:

A2U $CM_m > 0$ is a constant such that

$$CM_m := \max_{i=1,...N}\{CM_{m,i}\}.$$

A3U $C_{D,m} > 0$ is a constant such that

$$||D^\beta \mathcal{W}_i||_{L^\infty} \le \frac{C_{D,m}}{d_i^{|\beta|}}, \qquad |\beta| \le m, \, i = 1, ..., N \; .$$

Using **A2U** and **A3U** the following global result can be proved:

Theorem 4.5 *Assume* ***A1****,* ***A2U*** *and* ***A3U****; and let* $p \ge 1$, $l \le m$. *Then*

$$|\, u - \mathcal{ST}^m(u)\,|_{l,p} \le C_{m,l}\, h^{m+1-l} |\, u\,|_{m+1,p}\,, \quad \forall u : u \in W_p^{m+1}(\Omega),$$

where

$$C_{m,l} = (\#\{\alpha : |\alpha| = l\})^{1/p}\, M^{2/p} C(n,m)\, C_{D,m}\, CM_m\,.$$

5 APPLICATION TO *H-P* CLOUD FUNCTIONS

In h-p cloud methods (see,[6] for example) the vectorial space $\mathcal{F}^m$, which is defined by

$$\mathcal{F}^m = \left\{ v : \, v = \sum_{i=1}^{N} P_i^m \mathcal{W}_i, \; P_i^m \in \mathcal{P}_i^m, \; i = 1, ..., N \right\},$$

is utilized to solve elliptic PDEs in a Galerkin scheme over a Sobolev space $H^m(\Omega)$. If $u \in H^{m+1}(\Omega)$ is the exact solution of the boundary-value problem, by Céa's lemma,[2,3] the error is estimated by an expression such as

$$C \inf_{v \in \mathcal{F}^m} || \, u - v \, ||_{m,2} \, .$$

Since $\mathcal{ST}^m(u) \in \mathcal{F}^m$, the results above can be applied in order to obtain error estimates of the boundary-value problem.

The following Jackson-type inequalities follow from theorem 4.5 .

Corollary 5.1 *Assume* ***A1U*** *and* ***A2U****; and let m be an integer ≥ 0 and $p \geq 1$. Then,*

$$\sup_{u \in W_p^{m+1}(\Omega)} \inf_{v \in \mathcal{F}^m} \frac{| \, u - v \, |_{l,p}}{| \, u \, |_{m+1,p}} \leq C_{m,l} \, h^{m+1-l} \quad \forall \, 0 \leq l \leq m,$$

where

$$C_{m,l} = (\#\{\alpha : |\alpha| = l\})^{1/p} M^{2/p} (C(n,m) C_{D,m} \, CM_m).$$

5.1 Essential boundary conditions

We assume that $\partial\Omega$ is sufficiently smooth so that all the following arguments are valid. Assume we want to use $\mathcal{F}^m$ in a Galerkin scheme to solve a boundary value problem with homogeneous Dirichlet boundary condition on $\partial\Omega$. A major difficulty is the imposition of essential boundary conditions. Even if the approximating function satisfies the Kronecker-delta condition, it is useful to take profit of the high order of the approximation in the same way as in the p version of FEM. This goal can be achieved by computing an appropriate $H^{r-1/2}(\partial\Omega)$ projection operator.

Let $Q_\partial^N \subset Q^N$ be the set of *nodes* in the boundary. We reorder indexes so that

$$\{1, ..., N\} = \{1, ..., M\} \cup \{M + 1, ..., N\}$$

and $Q_\partial^N = \{\{1, ..., M\}$.

For $x_k \in Q_\partial^N$, let $\mathbf{n}_k$ be the normal vector of $\partial\Omega$ at x_k. We define the space of tangent Taylor polynomials at x_k by

$$\mathcal{P}_{\partial,i}^m := \{P \in \mathcal{P}_i^m : D_{\mathbf{n}_k}(P) = 0\}.$$

Now, the space $\mathcal{F}_\partial^m \subset H^{1/2}(\partial\Omega)$ is defined by

$$\mathcal{F}_\partial^m := \{v \in H^{1/2}(\partial\Omega) : v = \sum_{j=1}^{M} (Q_j \, \mathcal{W}_j) \,|\, \partial\Omega ; \, Q_j \in \mathcal{P}_{\partial,j}^m \,, j = 1, ..., M\}.$$

If $\{v_k\}_{k=1}^L$ is the natural basis of $\mathcal{F}_\partial^m$, we define the projection operator

$$p_\partial : H^r(\Omega) \to \mathcal{F}_\partial^m$$

by the formula

$$p_\partial(u) := \sum_{k=1}^{L} \left(\int_{\partial\Omega} (u|\partial\Omega) \, v_k \, dS \right) \, v_k.$$

It is not difficult to show that the restriction $p_{\mathcal{F},\partial} : \mathcal{F}^m \to \mathcal{F}_\partial^m$ is a surjective linear operator and that we have a canonical decomposition

$$\mathcal{F}^m = \mathcal{L} \oplus \mathcal{F}_0^m$$

where $\mathcal{F}_0^m = \ker(p_{\mathcal{F},\partial})$.

The decomposition can be used to prove:

Claim 5.2 *Assume* ***A1U, A2U*** *and $\partial\Omega$ is smooth. Let m be an integer ≥ 0, then*

$$\sup_{u \in H_0^{m+1}(\Omega)} \inf_{v \in \mathcal{F}_0^m} \frac{|\, u - v \,|_l}{|\, u \,|_{m+1}} \leq C_{m,l} \, h^{m+1-l} \quad \forall \, 0 \leq l \leq m,$$

where

$$C_{m,l} = (\#\{\alpha : |\alpha| = l\})^{1/2} M \, C(n,m) \, C_{D,m} \, C M_m.$$

We will not prove this claim in this work. A more detailed account of this approach is the subject of a forthcoming paper. Our aim here was to hint a way to handle essential boundary conditions satisfactorily. This scheme can be generalized to more general non homogeneous Dirichlet normal boundary value conditions.[12]

We assume this fact in the following subsection where we deal with another aspect of the problem of estimating errors.

5.2 Error estimates

We demonstrate our principal result here in connection with a model problem involving the following assumptions (see J. T. Oden[12]):

- We consider a regularly elliptic boundary value problem

$$\begin{aligned} Au &= f \quad \text{in } \Omega \\ B_j u &= g_j \text{ on } \partial\Omega \quad 0 \leq j \leq k-1, \end{aligned}$$

where A is a $2k$th-order elliptic partial differential operator with smooth coefficients, and $\{B_j\}_{j=0}^{k-1}$ is a normal system of boundary operators covering A. We consider a non homogeneous Dirichlet problem.

- The data $(f; g_j)$ are such that a solution $u^* \in H^r(\Omega)$ exists, with $r \geq 2k$. By (5.61) of,[12] there exists a constant $C > 0$ such that

$$|| u^* ||_r \leq C \left(|| f ||_{r-2k} + \sum_{j=0}^{k-1} || g_j ||_{\partial\Omega, r-q_j-1/2} \right). \tag{6}$$

- Let $B(u, v)$ be the bilinear form associated to this problem such that the variational equivalent problem is to find $u \in H^r(\Omega) \cap H_g^k(\Omega)$ such that

$$B(u, v) = l(v) \quad \forall v \in H_g^k(\Omega).$$

  $B(u, v)$ is assumed to be a continuous, weakly coercive, bilinear form on $H^k(\Omega)$ $\times H^k(\Omega)$, l is a linear functional on $H^k(\Omega)$ generated by $(f; g_j)$, and

$$H_g^k(\Omega) = \{v \in H^k(\Omega);\ B_j v = g_j \text{ en } \partial\Omega;\ 0 \leq j \leq k-1\}.$$

  It is well known that we can reduce the problem above to an homogeneous Dirichlet problem and still include the effects of non homogeneous boundary data. Consequently, we assume homogeneous Dirichlet conditions.

- Next, we construct a Galerkin approximation U^* of the problem, using the family $\mathcal{F}_0^m$, $k < m + 1 \leq 2k$. Let

$$e = u^* - U^*$$

  the approximation error.

- Finally, we assume **A1U** and **A2U.**

By Céa's lemma, there exists a constant $C > 0$ such that

$$|| e ||_k \leq C \inf_{U \in \mathcal{F}_0^m} || u^* - U ||_k \leq C || u^* - p_D(\mathcal{A}\mathcal{T}^m(u^*)) ||_k.$$

Then, since $|| u^* ||_{m+1} \leq || u^* ||_r$, (5.2) gives

$$|| e ||_k \leq \widetilde{C}\, d^{m+1-k} || u^* ||_r.$$

Finally, in view of (6)

$$|| e ||_k \leq \overline{C}\, d^{m+1-k} \left(|| f ||_{r-2k} + \sum_{j=0}^{k-1} || g_j ||_{\partial\Omega, r-q_j-1/2} \right).$$

This is an example of the FEM error estimates that we could obtain using the estimates proved in this work.

Remark 5.3 *The result above is in fact a theorem once that the problem of essential boundary conditions was handle and it is valid for natural boundary conditions.*

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, Inc., Orlando, 1975.

[2] S.C. Brener and L.R. Scott, *The mathematical Theory of Finite Elements Methods*, Springer-Verlag, New York, 1994.

[3] P. G. Ciarlet, *The Finite Elements Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[4] C. A. Duarte and J. T. Oden, *Hp clouds-a meshless method to solve boundary-value problems*, Technical Report 95-05, TICAM, The University of Texas at Austin, 1995.

[5] ______, *An h-p adaptive method using clouds*, Comput. Meth. Appl. Mech. Engrg. **139** (1996), 237–262.

[6] ______, *H-p clouds-an h-p Meshless Method*, Num. Meth. for Partial Diff. Eq. **1** (1996), 1–34.

[7] C. A. M. Duarte T. J. Liszka and W. W. Tworzydlo, *hp-meshless cloud method*, Comput. Method Appl. Mech. Engrg. **139** (1996), 263–288.

[8] R. G. Durán, *On polynomial approximation in Sobolev spaces*, SIAM J. Numer. Anal. **20** (1983), 985–988.

[9] M. Fleming T. Belyschko, Y. Krongauz and P. Krysl, *Meshless methods: an overview and recent depelopments*, Comp. Methods Appl. Mech. Engrg. **139** (1996), 3–48.

[10] W. Han and X. Meng, *Error analysis of the Reproducing Kernel Particle Method*, Comp. Meth. in Applied Mech. and Engrg. **190** (2001), 6157–6181.

[11] Shaofan Li W. K. Liu and T. Belytscho, *Moving Least Square Reproducing Kernel Methods. Part I: Methodology and convergence*, Comput. Methods in Appl. Mech. and Engrg. **143** (1997), 113–154.

[12] J. T. Oden and J. N. Reddy. *An Introduction to the Mathematical Theory of Finite Elements*, John Wiley.

[13] R. Verfúrth, *A note on polynomial approximation in Sobolev spaces*, RAIRO Modél. Anal. Numér. **33** (1999), 715–719.

[14] C. Zuppa, *Error estimates for modified local Shepard's formulas in Sobolev spaces*, RAIRO Modél. Anal. Numér. , submitted, 2003.

[15] C. Zuppa, *Jackson type inequalities for h-p clouds and error estimates for the penalty method*, to be published, 2003.