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ERROR ANALYSIS IN THE IMPLEMENTATION OF THE DUAL RECIPROCITY BOUNDARY ELEMENTS METHOD

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Abstract. *This work presents the formulation and numerical implementation of the dual reciprocity boundary element method applied to the analysis of elastostatic problems considering body loads. Provided the traditional boundary element method is unable to treat directly the influence of body loads, the dual reciprocity approach is used to make this consideration, transforming domain integrals in equivalent boundary integrals so that the method does not lose its most interesting features which are discretization and integration only on the boundary of the problem. The numerical results obtained from the analysis of a rotating disc are compared to the analytical solution for this problem in order to verify the influence of mesh refinement and internal node density on the accuracy of the method in study.*

1 INTRODUCTION

The computation of body forces in standard elastostatic analysis is an essential characteristic of modern numerical analysis methods. Applications like dynamic analysis, gravity forces and crack patch repairing need this kind of implementation. The Dual Reciprocity Boundary Element Method (DRBEM) is shown as an alternative way to treat body forces using the boundary integral equation approach. Besides the mathematical and numerical implementation, the main advantages of the dual reciprocity method are presented and the accuracy of results through an error analysis is discussed. In this first section, it is shown a theoretical discussion about the necessity for the use of the dual reciprocity method and the differences in comparison with the standard boundary element method.

The dual reciprocity method is presented as an alternative to the domain cell integration, used to treat the body forces in the boundary integral equation approach. This treatment is carried out by computing domain integrals, with techniques like Gauss method, inside finite cells created to discretize the domain where the body forces exist. Although it is a useful approach to take into account the body forces, provided it handles with the domain integral of body forces, it is not so interesting. The domain cell integration method reduces the attractive characteristic of the boundary element method that is the exclusive use of boundary integral equations evaluated on the boundary of the problem.

In order to keep the boundary integral equations and handle with the body forces, it is necessary to transform the domain integral, related with the body forces, into boundary integrals. The way to do this is the same way used to bring the domain integral equation, obtained from the Navier's elasticity equation of equilibrium, from the domain to the boundary of the problem. This procedure involves the application of the reciprocity theorem. After applying the this theorem to the complete domain integral it is transformed into boundary integrals, but it remains one domain integral, that is the one related with the body forces. Then, the reciprocity theorem is applied again to transform the body force domain integral in a boundary integral, which is able to represent the body forces. Provided the reciprocity theorem is applied twice, this method is called dual reciprocity, and it transforms the body force domain integral into equivalent boundary integrals approximating the body forces through interpolation functions.

Note that once the DRBEM formulation is not completely free from domain representations, it needs some domain co-ordinates to evaluate functions of its kernel. These domain co-ordinates are called domain nodes. The influence of the amount of these nodes and the boundary elements mesh refinement in the accuracy of the results obtained with the method is an important characteristic and will be discussed in this work.

2 FORMULATION

The DRBEM was first proposed by Nardini and Brebbia[1] (1982) and has been used by several authors through years with few modifications. The formulation used in this work is quite similar to the one used by Partridge, Brebbia and Wrobel[2] (1992) and by Domínguez[3] (1993).

The starting point for DRBEM formulation is the boundary integral equation[4] itself. This equation is obtained through the application of the reciprocity theorem on the equilibrium equation of the system, and is shown in equation 1.

$$\int_\Gamma t_i\, u^*_{ik} d\Gamma + \int_\Omega b_k u^*_{ik} d\Gamma = \int_\Gamma t^*_{ik} u_i \, d\Gamma + c_{ij} u_i \ \mathbf{(d)} \quad (1)$$

As mentioned before, there is a domain integral, equation 2, mixed between the boundary integrals, that is due to the body forces.

$$\int_\Omega u^*_{lk} b_k d\Omega \quad (2)$$

The b_k term is approximated by a set of coefficients and functions, as

$$b_k = \sum_{j=1}^{N+L} f^j \alpha^j_k \quad (3)$$

in this equation $\alpha^{(j)}_k$ represents a set of coefficients to be determined, initially unknown and the f^j represents the approximation functions based in the geometry of the problem.

There is a solution, $\hat{u}^{(j)}_{mk}$; capable to satisfy the Navier's equation like mentioned in Kane[5] (1993), which can be determined by equation 3.

$$G \cdot \hat{u}^j_{mk,ll} + \frac{G}{1-2\nu} \cdot \hat{u}^j_{lk,lm} = \delta_{mk} f^j \quad (4)$$

In order to solve the DRBEM problem it is necessary to determine an amount of $\hat{u}^{(j)}_{mk}$ solutions equal to the total number of nodes defined in the problem, or the number of nodes after summing the boundary nodes and the internal DRBEM nodes. These solutions are called particular displacements.

Substituting equations 3 and 4 in the domain integral shown in equation 2 and applying the reciprocity theorem to the remaining domain integral leads to:

$$c^i_{lk} u^i_k = \int_\Gamma u^*_{lk} t_k d\Gamma - \int_\Gamma t^*_{lk} u_k d\Gamma + \sum_{j=1}^{N+L} \left(c^i_{lk} \hat{u}^{ij}_{mk} + \int_\Gamma t^*_{lk} \hat{u}^j_{mk} d\Gamma - \int_\Gamma u^*_{lk} \hat{t}^j_{mk} d\Gamma \right) \quad (5)$$

in this equation 5, $\hat{t}^j_{mk}$ represents the traction related to the particular displacements $\hat{u}^j_{mk}$.

Equation 5 is free of domain integrals and is ready to be evaluated by the boundary element method. Note that the reciprocity theorem is applied twice, first to transform the whole domain integral equation into boundary integral equation and then to transform the remaining domain integral equation in to a boundary integral equation, leading to the equation 5. This procedure gives the name to the method.

Provided there is a final equation, what means boundary integrals only able to handle with body force through a boundary integral representation, it is possible to represent this equation in a matrix form, in order to prepare a system to a numerical implementation.

$$\mathbf{c}^i\mathbf{u}^i = \int_\Gamma \mathbf{u}^*\mathbf{t}\, d\Gamma - \int_\Gamma \mathbf{t}^*\mathbf{u}\, d\Gamma + \sum_{j=1}^{N+L}\left(\mathbf{c}^i\hat{\mathbf{u}}^{ij} + \int_\Gamma \mathbf{t}^*\hat{\mathbf{u}}^j d\Gamma - \int_\Gamma \mathbf{u}^*\hat{\mathbf{t}}^j d\Gamma\right) \tag{6}$$

The matrix system represented in equation 6 is ready for the application of a numerical evaluation method like the Boundary Element Method (BEM). In order to evaluate the analytical integration through this numerical method, it is necessary to transform them into ordinary summations, given by

$$c^i u^i + \sum_{k=1}^{N} \overline{H}_{ik} u_k - \sum_{k=1}^{N} G_{ik} t_k = \sum_{j=1}^{N+L}\left(c^i \hat{u}^{ij} + \sum_{k=1}^{N} \overline{H}_{ik} \hat{u}_k^j - \sum_{k=1}^{N} G_{ik} \hat{t}_k^j \right)\cdot \alpha^j \tag{7}$$

It is interesting to see that the equation 7 is obtained after the application of the boundary element method, in other words, a discretized boundary element mesh approximates the continuum. This means that all the continuum values, related with physical or geometrical properties, are represented by interpolation functions called shape functions. Note that the values of $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{t}}$ are known, provided they are dependent only on geometric data of the elements and internal nodes, although shape functions are used to interpolate their values in the same way that are used to interpolate ***u*** and ***t***. This procedure brings a simplification in the numerical implementation because it makes possible to use the same H and G matrices in both sides of the equation 7. Although, this technique introduces an error in the evaluation of the right hand side terms of equation 7, Partridge, Brebbia and Wrobel[2] (1992) have shown that this error is worthless and there is a significant improvement in the efficiency of the method.

At this point, equation 7 is ready to be used. In order to generate a matrix equation system it is necessary to apply equation 7 to each node of the problem, what leads to the general matrix equation shown in equation 8.

$$\boldsymbol{H}\cdot\boldsymbol{u} - \boldsymbol{G}\cdot\boldsymbol{t} = \left(\boldsymbol{H}\cdot\hat{\boldsymbol{U}} - \boldsymbol{G}\cdot\hat{\boldsymbol{T}}\right)\cdot\boldsymbol{\alpha} \tag{8}$$

It is possible to make $\boldsymbol{\alpha} = \boldsymbol{F}^{1}\boldsymbol{b}$, generating

$$\boldsymbol{H}\cdot\boldsymbol{u} - \boldsymbol{G}\cdot\boldsymbol{t} = \left(\boldsymbol{H}\cdot\hat{\boldsymbol{U}} - \boldsymbol{G}\cdot\hat{\boldsymbol{T}}\right)\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{b} \tag{9}$$

equation 9 is the basic equation to use the DRBEM and involves just boundary integral equation.

As it can be seen, there are some new definitions typically used by DRBEM that are not easily understandable by conventional BEM users. Definitions like internal nodes, α vector and approximation functions will be better explained in the section below.

3 DRBEM DEFINITIONS – INTERNAL NODES

The first DRBEM definition is the internal node concept. Conventional BEM uses boundary nodes, that represents characteristic points on the boundary geometry of the problem needed to define the limits of the elements. Then, boundary nodes usually are associated with the boundaries of every single element on the surface of the problem. In other way, internal nodes are typically used by DRBEM and are significantly different of the boundary nodes. First, the internal nodes are defined in the domain of the problem. Second, they do not represent characteristic points of the geometry and are not used to define limits of any kind of element.

The internal nodes are defined in order to represent domain characteristics. Note that it is not necessary to define internal node in order to obtain the boundary solution, although the accuracy of the solution is strongly related with the amount of internal nodes used when there is a body load involved. The sufficient number of internal nodes will be discussed later in this paper.

Besides the solution of the problem, internal nodes can be used like conventional BEM internal points in order to have domain values needed to improve solution interpolation and post-processing colour map generation. In some kind of problems, such as the one used in this work, it is possible to obtain the internal node solution directly, shown by Partridge, Brebbia and Wrobel[2] (1992). In this case, it is possible to evaluate equation 7 inside the domain, what means that the value of the variable ***c*** is *1*. Then, equation 7 can be rewritten like equation 10.

$$\boldsymbol{u}^{\boldsymbol{i}} = -\sum_{\boldsymbol{k=1}}^{\boldsymbol{N}} \overline{\boldsymbol{H}}_{\boldsymbol{ik}} \boldsymbol{u}_{\boldsymbol{k}} + \sum_{\boldsymbol{k=1}}^{\boldsymbol{N}} \boldsymbol{G}_{\boldsymbol{ik}} \boldsymbol{t}_{\boldsymbol{k}} + \sum_{\boldsymbol{j=1}}^{\boldsymbol{N+L}} \left(\boldsymbol{c}^{\boldsymbol{i}} \hat{\boldsymbol{u}}^{\boldsymbol{ij}} + \sum_{\boldsymbol{k=1}}^{\boldsymbol{N}} \overline{\boldsymbol{H}}_{\boldsymbol{ik}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}^{\boldsymbol{j}} - \sum_{\boldsymbol{k=1}}^{\boldsymbol{N}} \boldsymbol{G}_{\boldsymbol{ik}} \hat{\boldsymbol{t}}_{\boldsymbol{k}}^{\boldsymbol{j}} \right) \cdot \alpha^{j} \tag{10}$$

All variables in this equation are known provided they were calculated by the solution of the problem. Only the variables related to the displacement of internal nodes remain unknown and can be calculated.

4 DRBEM DEFINITIONS – α VECTOR

The α vector can be obtained by evaluating equation 3. Just to make it easy to understand, equation 3 will be reproduced and is now presented as equation 11.

$$b_k = \sum_{j=1}^{N+L} f^j \alpha_k^j \tag{11}$$

Since it is necessary to obtain the α vector, the other components of the equation 11 must be known. The other components of equation 11 are the approximation functions, that will be assumed as known functions provided they will be chosen from a variety of proposed ones, and the ***b*** vector. The ***b*** vector is related with the body forces evaluated in the domain of the problem. This body force can be calculated using different equations depending on the kind of body force that it is necessary to take into account. In the case of body force due to inertia in a rotating disk, with angular velocity ω and density ρ, the ***b*** vector on a point x_i is given by

$$b_i = \rho\omega^2 x_i \tag{12}$$

other kinds of body forces like domain loads or magnetic fields will need different mathematical functions in order to give appropriated ***b*** vectors.

Equation 11 can be written in matrix form in order to prepare it to be solved by a mathematical method. Then, equation 11 can be written as

$$\mathbf{b} = \mathbf{F}\boldsymbol{\alpha} \tag{13}$$

In equation 13, **b** is a column vector; each line containing one body force evaluated according to a body force relation in each one of the DRBEM collocation points. **F** is a matrix; each column containing a f_i vector of approximation functions evaluated in every collocation point of the DRBEM.

Since all the members of equation 13 are known, it is possible to obtain the α vector by solving the matrix system.

$$\boldsymbol{\alpha} = \mathbf{F}^{-1}\mathbf{b} \tag{14}$$

Then, a ***d*** vector of known values can be substituted in the right hand side of equation 9, resulting in

$$\boldsymbol{H\,u} - \boldsymbol{G\,t} = \boldsymbol{d} \tag{15}$$

with ***d*** vector defined by equation 16.

$$\boldsymbol{d} = (\boldsymbol{H}\,\hat{\boldsymbol{U}} - \boldsymbol{G}\,\hat{\boldsymbol{T}})\,\boldsymbol{\alpha} \tag{16}$$

the ***d*** vector introduces the body force influences and it is obtained by multiplying known matrices and vectors.

5 DRBEM DEFINITIONS – APPROXIMATION FUNCTIONS

The approximation functions, f_i, and the particular solutions $\hat{u}$ and $\hat{t}$, used in the DRBEM, have not been defined by the formulation. The unique restriction for them is that the resulting matrix **F**, equation 13, can not be singular.

In order to define the approximation functions, it is usual to propose a mathematical relation for f and calculate the $\hat{u}$ and $\hat{t}$ based on it using equation 4. The mathematical relations commonly proposed for the approximation functions are: trigonometric series; Pascal's triangle elements; and the r distance used to define the fundamental solutions.

In this work it will be used a function known as r type function, that was first used by Nardini and Brebbia[1] (1982) and adopted by the majority of the researchers after them due to its simplicity and accuracy. This function is based on the series shown in equation 17.

$$f = 1 \pm r \pm r^2 \pm \ldots \pm r^m \tag{17}$$

Theoretically, any kind of combination of the terms of equation 17 may be used as r type approximation function. In this work it will be used the function

$$f(r) = 1 - r \tag{18}$$

This combination has been chosen because of the good results shown by Domínguez[3] (1993). This kind of function is interesting because it assures the non singularity of the **F** matrix since the main diagonal of the matrix will never be zero.

6 FINAL DRBEM FORMULATION

The definition of the approximation function to be used in the DRBEM formulation leads to a particular form of the general formulation shown before. The difference occurs in the evaluation of the $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{t}}$ functions, since both of them depend on the kind of approximation functions chosen.

In the case of an approximation function like the one shown in equation 18, substituting it in the equation 4, the $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{t}}$ functions will be expressed by the equations 19 and 20.

$$\hat{u}_{kn} = \frac{1}{G} \cdot \left\{ \left[\frac{1-2\nu}{5-4\nu} + \frac{r}{30\cdot(1-\nu)} \right] \cdot r^2 r_{,k} r_{,n} - \frac{9-10\nu}{90\cdot(1-\nu)} \delta_{kn} r^3 \right\} \tag{19}$$

$$\begin{aligned} \hat{t}_{kn} = \Bigg\{ & \frac{2\nu}{1-2\nu} \cdot \left(3A \cdot r + 4B \cdot r^2 + 3D \cdot r^2\right) \cdot r_{,n} \delta_{kj} + \Big[2 \cdot \left(A \cdot r + B \cdot r^2\right) \cdot r_{,n} \delta_{kj} + \\ & + \left(A \cdot r + B \cdot r^2 + 3D \cdot r^2\right) \cdot \left(r_{,k} \delta_{nj} + r_{,j} \delta_{kn}\right) + 2B \cdot r^2 r_{,k} r_{,n} r_{,j} \Big] \Bigg\} \cdot n_j \end{aligned} \tag{20}$$

with: $A = \frac{1-2\nu}{5-4\nu}$; $B = \frac{1}{30\cdot(1-\nu)}$; $D = \frac{10\nu - 9}{90\cdot(1-\nu)}$, G is the shear modulus and ν is the Poisson ratio.

7 COMPUTATIONAL IMPLEMENTATION

The numerical implementation[6] of DRBEM has been made using a pre-existent BEM base program, developed to solve 2D, elastostatic, isotropic and linear problems using boundary element methods. In order to implement the DRBEM on this program it was necessary to add new functions specifically to evaluate the new terms introduced by the DRBEM. No modifications were needed in the basic BEM code, except some verification keys to change the program flux between conventional BEM and DRBEM.

The function calc_alfa is responsible for calculating the α vector; calc_UcPc is used to obtain the particular solution matrices $\hat{U}$ and $\hat{T}$ using the functions calc_uchap and calc_pchap in order to obtain the numerical values for the particular solutions. Finally, the function calc_d receives the results of all previous functions and calculates the $\boldsymbol{d}$ vector used by DRBEM to take into account the body force influence.

Although the program was implemented to compute the rotating disk inertial terms, the implementation can be easily extended to other domain forces with little changes in the calc_alfa function. This facility is possible in the majority of the cases since the inertial terms do not depend on node variables like displacement or tractions. The easily extention procedure is due to the modularization of the DRBEM functions which separates them from the main BEM functions.

The data flow of the program[6] is shown in figure 1.

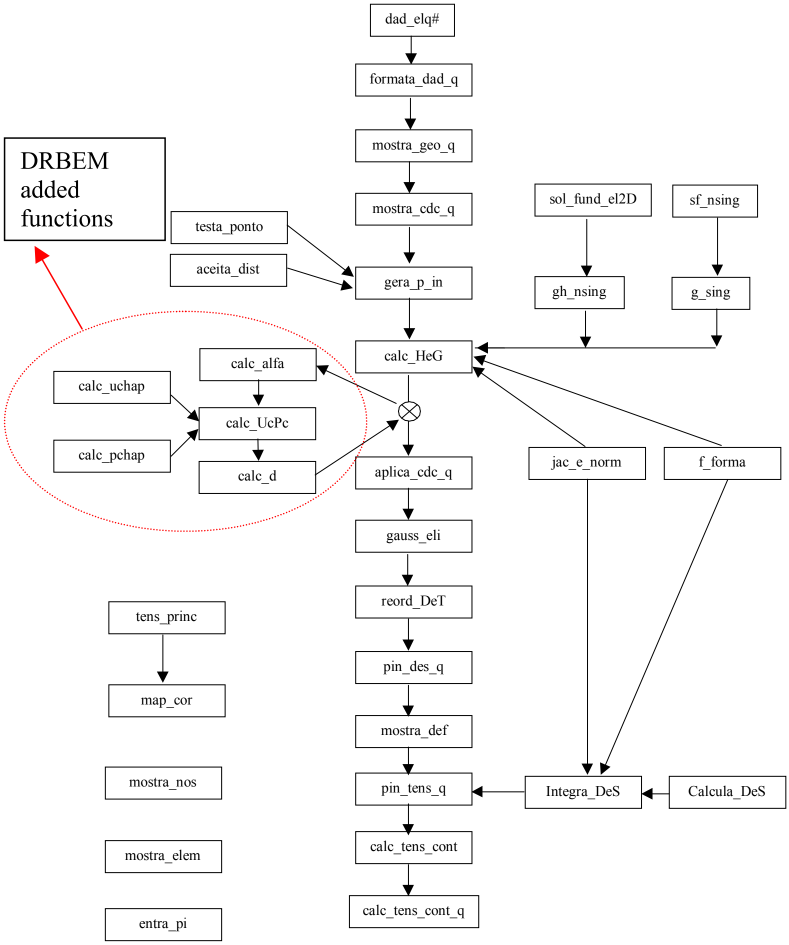

Figure 1. DRBEM program data flow[6]

8 RESULTS

In order to obtain the influence of parameters like amount of internal nodes and boundary element mesh refinement in the accuracy of the DRBEM, it will be analysed a well known problem, which has analytical solution that makes possible to compare the numerical method result with an analytical benchmark.

The reference problem is a rotating disk, which has analytical solution for the radial displacement u_r, shown by Tymoshenko[7] (1970) and is given by

$$u_r = \frac{1}{E} \cdot \left[\frac{1-\nu}{4} \left(\rho \omega^2 r^3 \right) \right] \tag{21}$$

Using this equation, it is possible to obtain the radial displacement u_r, provided the Young's modulus E, the Poisson ratio ν, the mass density ρ, the angular velocity ω and the rotating disk radius r.

Since it is known the analytical solution for the benchmark problem, it is necessary to solve it through the numerical method, in this case the DRBEM. Like in any other BEM problem, it is necessary to model the boundary geometry of the problem and choose the boundary element mesh to be applied to this geometry. Due to the intrinsic symmetry of this kind of problem, it has been modelled just a quarter of the rotating disk and symmetric boundary element conditions were applied to the symmetry lines. The geometry and initial boundary element mesh can be seen in figure 2.

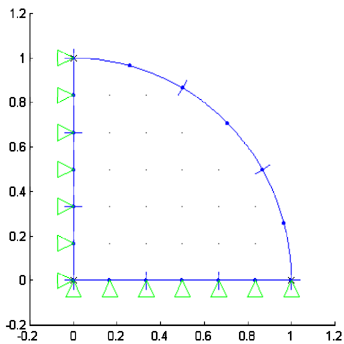

Figure 2. Symmetric model of rotating disk to DRBEM analysis.

It is possible to see in figure 2 a set of internal nodes regularly distributed through the domain. These nodes were automatically generated by the code and used in the DRBEM calculation. Both, internal nodes and boundary element mesh can be automatically generated by the code in order to change the mesh and internal node quantity and distribution. So, the same problem can be analysed many times, using different kind of internal node distribution and different boundary element mesh discretization at a time.

After this process, calculated data can be grouped conveniently to create a convergence surface graphic. This surface shows the error between the analytical solution and the DRBEM solution for each pair, boundary element mesh and internal point quantity. Then, the convergence surface makes possible to analyse the DRBEM solution behaviour according to the internal nodes and boundary element mesh variation.

The percentage error is calculated by comparing the analytical solution, evaluated according equation 21, with the respective DRBEM numerical method solution.

The problem data, necessary to evaluate the analytical solution and to apply the DRBEM analysis, are given in table 1.

Table 1. Material properties and problem data

Young's modulus (E)	71000 MPa
Poisson ration (ν)	0,3
Material mass density (ρ)	8000 kg/m^3
Rotating disk radius (r)	1 m
Rotation speed (ω)	100 rad/s

Since these data are substituted in equation 21, the analytical solution for the rotating disk can be numerically evaluated as 1.972×10^{-4} m, then, the benchmark is set to this value.

Using the same set of data, shown in table 1, it is possible to run the DRBEM program and create different discretizations for the mesh and different amount of internal nodes. These changes result in different solution values that can be compared with the analytical benchmark.

In order to create the convergence surface, boundary element meshes have been created from a minimum discretization of one (1) boundary element per boundary edge to a maximum discretization of ten (10) boundary elements per boundary edge. For each one of this boundary element meshes it has been created different amount of internal nodes from a minimum of zero (0) internal node to a maximum of sixty-four (64).

The resulting convergence surface can be seen in figure 3. The convergence behaviour can be better analysed using table 2.

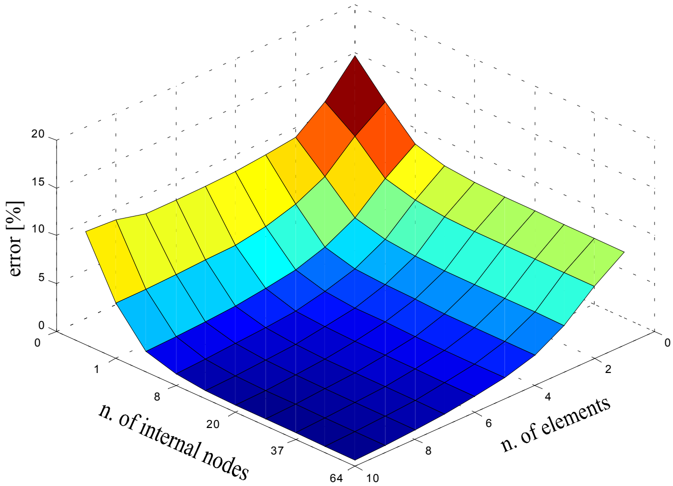

Figure 3. Convergence analysis surface

Table 2. Percentual error behaviour according internal nodes and amount of elements variation.

n. of elements / n. internal nodes	1	2	3	4	5	6	7	8	9	10
0	17,6	14,1	11,8	11,4	11,1	10,9	10,9	10,9	11,7	11,8
1	14,1	11,9	9,2	7,7	6,8	6,2	6,0	5,9	5,8	5,8
4	11,1	8,9	6,2	4,6	3,6	3,1	2,7	2,4	2,3	2,2
8	10,3	8,1	5,4	3,8	2,8	2,2	1,8	1,5	1,3	1,2
13	10,0	7,8	5,1	3,5	2,5	1,9	1,5	1,2	1,0	0,9
20	9,9	7,6	5,0	3,3	2,4	1,8	1,4	1,1	0,9	0,8
28	9,8	7,6	4,9	3,3	2,3	1,7	1,3	1,1	0,8	0,7
37	9,8	7,5	4,9	3,2	2,3	1,7	1,3	1,0	0,8	0,7
50	9,8	7,5	4,9	3,2	2,3	1,7	1,3	1,0	0,8	0,7
64	9,8	7,5	4,9	3,2	2,3	1,7	1,3	1,0	0,8	0,6

From figure 3 or table 2 it is possible to see that increasing the boundary element mesh discretization leads to a reduction in the error until a stable level near to 10%, without using internal nodes. On the other hand, for a minimum boundary element mesh, it is possible to see that the increasing of internal node quantity also leads to a reduction in the error until a stable level near of 10%. So, the simple procedure of refining the boundary element mesh is not sufficient when it is necessary to take into account the body force effect in order to solve problem. Similarly, increasing the amount of internal nodes for DRBEM solution, using a poor boundary element mesh, produces as good results as the boundary element mesh refinement by itself.

Analysing the convergence surface it is possible to see a lower region, less than 1% error, that could be a good region to work for DRBEM solution. This low error region is a combination of a representative boundary element mesh refinement and a quantity of internal nodes able to well consider the body forces.

So, in the rotating disk case, using around eight (8) elements per boundary edge and twenty (20) internal nodes can make a DRBEM representative analysis of the problem.

9 CONCLUSIONS

The accuracy of DRBEM solution is strongly dependent on the boundary element mesh refinement and on the amount of internal nodes used to obtain the solution for the body force problem. This characteristic could be analysed by the convergence surface shown in figure 3.

The identification of such behaviour is interesting because it makes possible to choose a minimum of both, boundary element mesh discretization and a number of internal nodes used to describe the geometry of the problem and its body force contribution. Besides, the knowledge of the influence of boundary element mesh refinement and the amount of internal nodes in the accuracy of the solution of the problem can help in reducing unnecessary computational work, reducing the time to solution of DRBEM solver.

This optimisation can be done provided the minimum boundary element mesh discretization and the less internal node quantity for a good accuracy of the solution. Reducing the amount of boundary elements and internal node leads to a reduction in the size of matrices needed to solve the problem. This kind of reduction leads to a reduction in the computational resource consumption and in the time to solution.

Then, the study of DRBEM convergence behaviour is a useful approach in order to obtain good accuracy of results with less computational resource and time consumption, which is an important concern when dealing with huge problems or more complex applications, like iterative body force solution.

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