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FINITE ELEMENT TEMPLATES: A SUMMARY OF RECENT DEVELOPMENTS

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1. INTRODUCTION

This lecture highlights recent accomplishments of a line of research in basic technology of the finite element method (FEM) pursued by the author at the University of Colorado over the past six years. The research began in 1987 with modest goals: to develop high-performance plate and shell finite elements with unconventional techniques, in particular the Free Formulation and the Assumed Natural Strain method.

These approaches, although successful, did not fit the standard variational framework of canonical functionals. Efforts to "varationalize" success led to a number of discoveries discussed below. As of this writing the next logical step of this research is the orderly production of finite element templates (defined below) as well as further development of a theoretical foundation that explains and justifies their existence. The application of templates to element-level error estimation also appears promising but will also require a sounder theoretical basis.

A finite element template, or simply *template*, is a parametrized algebraic form that yields a continuum of convergent finite elements of fixed type. Here by "type" is meant an element selected for a specific application and with a given degree-of-freedom (dof) configuration; for example a 3-node, 9-dof Kirchhoff plate bending triangle. A form that yields all convergent elements of a given type is called an *universal template*. A form that yields a practically useful subset is called a *generic template*, where "generic" is used in the biological sense of "pertaining to a genus."

Obtaining an explicit universal template may be viewed as "closing the book" on an element type once and for all. For multidimensional elements, however, universal templates can become too complex or lack physical transparency. In such cases a generic template may represent a viable compromise.

2. MOTIVATION

Why are templates worth studying? Four reasons, taken from a recent survey paper [1] may be offered.

(i) The confusion associated with the vast number of elements claimed in the literature clears up: such elements become merely "points" in a parametric continuum. This important point is further discussed in Section 7.

(ii) Construction and testing of parametrized finite-element libraries can be simplified.

(iii) Another realm of mesh adaptivity opens up: c or t adaptation (the prefix is not fully decided as of this time), in which programs can pick up elements from the parametrized continuum to strive for maximum accuracy on a *fixed* mesh.

(iv) Several persistent mysteries may be clarified: convergence, mixability, accuracy, locking, spurious modes, and distortion sensitivity. With further theoretical advances such properties could be traced to components of the template and interpreted physically.

From a FEM teacher's viewpoint, (i) is particularly endearing. Unification has the potential of streamlining the teaching of finite element theory, and hence release precious instruction time to cover problem modeling aspects rather than spending class after class going over the "finite element catalog." From a FEM implementor's viewpoint, (ii) and (iii) are clearly most important. From a problem modeler's viewpoint, (iv) is of interest as a reliable way of avoiding surprises.

Aside from these specific concerns, the unification of all possible elements of given type is appealing from an aesthetic standpoint. Developing an individual finite element with the usual techniques, essentially unchanged since 1970, can take weeks or months; or even years if nonlinear and dynamic applications are pursued. The end product, however, takes merely a "potshot" at the infinite number of element instances.

3. THE EXPANDING FEM UNIVERSE

At the time of the author's doctoral work (1965-66) the finite element method was still unsettled, although the "core view" was rapidly solidifying. In that view the Finite Element Method (FEM) is regarded as a subset of the Ritz-Galerkin approximation procedure with piecewise-polynomial conforming basis functions of local support. The theoretical basis for this interpretation is sound, well developed and presumably approaching a final form.

Forsaking this narrow but safe niche incurs risks but promises rewards. As usual the aphorism "no pain, no gain" applies. Reward opportunities include increasing modeling and implementation flexibility, potential for improved performance, and unification with other approximation techniques. Risks relate to the lack of a sound theoretical foundation. Experience has shown that theory may take decades to develop and solidify, forcing adventurous researchers to rely on intuition and experimentation. Moreover, theory is often spurred and shaped by successful heuristic advances.

Risks and rewards increase with distance from the core. In its vicinity one deals with minor departures from the Ritz-Galerkin canon exemplified by mixed principles and "variational misdemeanors" such as numerical integration. Further away lie less understood devices: hybrid principles, nonconforming shape functions, kinematic tricks and other variational crimes. Finally one reaches the largely uncharted domain of the *purely algebraic approach*, where shape functions have disappeared, variational principles decline in importance, and the FEM blends naturally with other methods such as finite differences and lumped-parameter (lattice, Hrenikoff) models.

4. ESCAPING THE RITZ-GALERKIN CORE

An important tool to escape the safe but confining FEM core has been the patch test. Originally developed by Bruce Irons to heuristically explain the erratic behavior of nonconforming plate elements [2], it gained publicity and respectability through the Strang-Fix monograph [3]. Following some perplexing behavior noted during the 1970s, the test was put in a firmer ground by Taylor, Simo, Zienkiewicz and Chan [4].

Despite its virtues the conventional multielement patch test is and will remain a verification device: the element has to be fully developed before it can be tested on a patch. The Individual Element Test (IET), proposed in 1975 by Bergan and Hanssen [5], does not suffer from that deficiency. An underlying goal of their development was to establish a test that could be directly carried out on the stiffness equations of a single element — an obvious improvement over the multielement form. In addition the test was to be constructive, *i.e.*, used as a guide during element formulation, rather than as a post-facto check. Relationships between the IET and the forms A, B and C of the patch test that were set out by Taylor *et al.* [4] are discussed in a recent article [6].

The constructivity goal was achieved a decade later when the rules of the Free Formulation (FF) were presented by Bergan and Nygård [7]. The key point of the FF is the decomposition of the element stiffness matrix into a basic and a higher order part:

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_h. \tag{1}$$

The basic stiffness $\mathbf{K}_b$ is responsible for verification of the IET and hence takes care of consistency, but is generally rank deficient. The higher order stiffness $\mathbf{K}_h$ makes $\mathbf{K}$ attain the proper rank and is responsible for stability and accuracy. Incompatible displacement assumptions used in this formulation only affect $\mathbf{K}_h$.

The involvement of the author with the FF began in 1984, when joint work with Bergan resulted in the development of two high-performance triangles for membrane and plate-bending analysis [8,9]. The higher order stiffness of these elements was scaled so (1) becomes $\mathbf{K} = \mathbf{K}_b + \beta\mathbf{K}_h$, where the positive factor β was used to improve behavior on coarse meshes through energy-balance techniques. The resulting Scaled Free Formulation (SFF) represents an early instance of the use of parameters to attain better performance.

Another technique for development of high performance elements through direct strain interpolation was pursued in the early 1980s by several investigators, notably Bathe [10], Hinton [11] and Park [12]. The approach was labeled Assumed Natural Strains or ANS by Park and Stanley [12]. The end products were plate and shell elements of high performance despite their simplicity.

5. PARAMETRIZED VARIATIONAL PRINCIPLES

Neither SFF nor ANS initially conformed with a conventional variational framework. For the SFF the "fitting" task was undertaken by the author in 1987. This was spurred by criticism from a reviewer of [8], who justly observed that conclusions on the impressive performance of the element were largely based on numerical results rather than theory. After several trials the SFF was shown to be associated with a multifield variational principle with one free parameter (the higher order stiffness scaling factor β) that "interpolated" between hybrid versions of the Potential Energy and Hellinger-Reissner principles [13].

A variational justification of a particular case of the ANS formulation was obtained subsequently by Militello and the author [14]. This variant was then reworked into the Assumed Natural Deviatoric Strain (ANDES) formulation. ANDES was associated with another one-parameter hybrid multifield functional that interpolated between the Potential Energy and Hu-Washizu functionals. A comparison of the SFF and ANDES functionals led to a general 3-parameter reformulation of the classical elasticity functionals [15,16]. Principles associated with such functionals have been labeled Parametrized Variational Principles, or PVPs. Subsequently multifield PVPs have been constructed for incompressible solids [17], micropolar elasticity [18] and linear electrodynamics [19].

6. THE TEMPLATE CONJECTURE

To date the main application of PVPs to computational mechanics has been the construction of high-performance finite elements (HPFEs). These are simple elements that can deliver engineering accuracy with coarse discretization. A list of pertinent publications to date is provided in the survey article [1].

Some recurring features were observed during the development of specific HPFEs. In particular all linear elements so far constructed fit the stiffness decomposition (1) in which the basic and higher-order components take on the congruential matrix forms

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_h = c\mathbf{L}\mathbf{E}\mathbf{L}^T + \beta\mathbf{T}_h^T\mathbf{S}\mathbf{T}_h. \tag{2}$$

Here c is the inverse of the element volume, area or length, β is a positive scaling factor, $\mathbf{L}$ and $\mathbf{T}_h$ are geometric matrices, $\mathbf{E}$ is the matrix of elastic moduli, and $\mathbf{S}$ depends on the geometry and constitutive behavior as well as the PVP in use. In addition to β, matrices $\mathbf{L}$, $\mathbf{T}_h$ and $\mathbf{S}$ may contain free parameters. Some pertain to the generating PVP while others do not. Parameters in $\mathbf{L}$, which affect the basic stiffness $\mathbf{K}_b$, should be the same for all elements in an assembly for otherwise the IET would fail. On the other hand, parameters affecting the higher order stiffness $\mathbf{K}_h$ may change from element to element without impairing convergence as long as positiveness and proper rank is maintained.

The template conjecture asserts that (2) is the most general form for all convergent linear elements of a given type (where "type" is defined in Section 1). The conjecture is readily proven in the case of simple one-dimensional elements such as bars or beams, for which universal templates are easily constructed [1]. But it is far from obvious for multidimensional elements.

Regardless of whether the conjecture on the universality of (2) is proved or disproved in the future, two interesting consequences of the template concept are discussed next: element fingerprinting and direct algebraic construction of $\mathbf{K}$.

7. ELEMENT FINGERPRINTING

Consider a universal or generic template that depends on a set of free parameters. The template represents an infinite number of elements. Each set of values yields an element instance. Some of these instances may agree with "name" elements already published in the finite element literature. (Of course the probability of hitting such elements is negligibly low should the values be selected randomly.) Therefore the parameter set may be viewed as a "fingerprinting code" that identifies elements uniquely.

Sometimes the same element instance has been published two or more times under different names. If the underlying formulations are not similar those duplications may be hard to catch. Fingerprinting via templates, however, can resolve such identity puzzles without ambiguity.

8. THE DIRECT ALGEBRAIC APPROACH

It is legitimate to view (2) as a parametrized matrix form to be generated according to a set of rules. These rules can be classified into essential and convenient. Essential rules insure that the element passes the IET and that the higher order stiffness is orthogonal to constant-stress states. Convenient rules include things such as enforcement of geometric invariance and use of deviatoric higher order strains.

The point to be stressed is that the rules may be applied as recipes without regard to the usual FEM paraphernalia: shape functions, variational principles, weighted residual arguments, etc. At this stage we have reached the vast expanse of the *purely algebraic approach* while dropping excess baggage along the way. For many elements the utter simplicity of the finite difference approach is regained. And simplicity is beauty.

In the Introduction of their seminal 1975 paper Bergan and Hanssen [5] write:

> "An important observation is that each element is only represented by the numbers in its stiffness matrix during the analysis of the assembled system. The origin of these stiffness coefficients is unimportant to this part of the solution process."

This vision proved elusive because the direct construction of the entries of $\mathbf{K}$ without benefit of (2) is actually a problem in nonlinear constrained optimization, which for but the simplest elements leads to intractable matrix Riccati equations. If thus posed the problem becomes much harder to tackle than through the familiar element construction methods. The template decomposition (2) and the concept of parametrization combine to make a rule-based algebraic approach practically feasible.

9. ELEMENT-LEVEL ERROR ESTIMATION

The emergence of PVPs has suggested an innovative approach to *a-posteriori* error estimation of computed finite element solutions. Let v denote the element node displacement vector retrieved from that solution. The quadratic form

$$U_h = \tfrac{1}{2}\mathbf{v}^T \mathbf{K}_h \mathbf{v}, \tag{3}$$

has the meaning of higher-order energy (HOE) absorbed by the element. If the exact solution of the problem consists of a constant stress state, this value vanishes over each element. Similarly, as the solution converges on account of mesh refinement, the element energy is increasingly dominated by the basic energy $U_b = \frac{1}{2}\mathbf{v}^T\mathbf{K}_b\mathbf{v}$.

This heuristic argument shows that U_h may be regarded as a *local error indicator*: a number that goes to zero in each element as the solution converges. Because only individual element information is required in (3), the HOE indicator is said to be *element level*. Such indicators enjoy computational and logistic advantages over those that require access to adjacent element information.

The connection of this concept to the theory of so-called Invariant Parametrized Variational Principles (IPVPs) is outlined in the survey paper [1], where references to pioneer work that make use of this new idea to drive mesh adaptation processes may be found.

10. CONCLUDING REMARKS

The research path that starts from the IET and ends with templates is covered in a article sequence. The first article [20], which describes the evolution of the past test, has been completed and accepted for publication. Sequel articles describing the direct algebraic approach to templates and element-level error estimation are in preparation.

REFERENCES

1. C. A. Felippa, A survey of parametrized variational principles and applications to computational mechanics, *Comp. Meth. Appl. Mech. Engrg.*, **113**, 1194, pp. 109–139.

2. B. M. Irons and A. Razzaque, Experiences with the patch test for convergence of finite elements, in *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Ed. by A. K. Aziz, Academic Press, New York, 1972, pp. 557–587.

3. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall. Englewood Cliffs, N.J.,1973.

4. R. L. Taylor, J. C. Simo, O. C. Zienkiewicz and A. C. Chan, The patch test: a condition for assessing FEM convergence, *Int. J. Numer. Meth. Engrg*, **22**, 1986, pp. 39–62.

5. P. G. Bergan and L. Hanssen, A new approach for deriving 'good' finite elements. MAFELAP II Conference, Brunel University, 1975; in *The Mathematics of Finite Elements and Applications – Volume II*, ed. by J. R. Whiteman, Academic Press, London, 1976, pp. 483–497.

6. C. Militello and C. A. Felippa, The individual element patch revisited, in *The Finite Element Method in the 1990's*, ed. by E. Oñate, J. Periaux and A. Samuelsson, CIMNE, Barcelona and Springer-Verlag, Berlin, 1191, pp. 554–564.

7. P. G. Bergan and M. K. Nygård, Finite elements with increased freedom in choosing shape functions, *Int. J. Num. Meth. Engrg.*, **20**, 1984, pp. 643–664.

8. P. G. Bergan and C. A. Felippa. A triangular membrane element with rotational degrees of freedom, *Comp. Meth. Appl. Mech. Engrg.*, **50**, 1985, pp. 25–69.

9. C. A. Felippa and P. G. Bergan, A triangular plate bending element based on an energy-orthogonal free formulation, *Comp. Meth. Appl. Mech. Engrg.*, **61**, 1987, pp. 129–160.

10. K. J. Bathe and E. N. Dvorkin, A four-node plate bending element based on Mindlin-Reissner plate theory and a mixed interpolation. *Int. J. Numer. Meth. Engrg.*, **21**, 1985, pp. 367–383.

11. H. C. Huang and E. Hinton, A new nine node degenerated shell element with enhanced membrane and shear interpolation, *Int. J. Numer. Meth. Engrg.*, **22**, 1986, pp. 73–92.

12. K. C. Park and G. M. Stanley, A curved C^0 shell element based on assumed natural-coordinate strains, *J. Applied Mech.*, **53**, 1986, pp. 278–290.

13. C. A. Felippa, Parametrized multifield variational principles in elasticity: II. Hybrid functionals and the free formulation, *Comm. Appl. Numer. Meth.*, **5**, 1989, pp. 79–88.

14. C. Militello and C. A. Felippa, A variational justification of the assumed natural strain formulation of finite elements: I. Variational principles, *Computers & Structures*, **34**, 1990. pp. 431–438.

15. C. A. Felippa and C. Militello, Developments in variational methods for high performance plate and shell elements, in *Analytical and Computational Models for Shells*, CED Vol. 3, ed. by A. K. Noor, T. Belytschko and J. C. Simo, The American Society of Mechanical Engineers, ASME, New York, 1989, pp. 191–216.

16. C. A. Felippa and C. Militello, Variational formulation of high performance finite elements: parametrized variational principles, *Computers & Structures*, **36**, 1990, pp. 1–11.

17. C. A. Felippa, Parametrized variational principles encompassing compressible and incompressible elasticity, *Internat. J. Solids Structures*, **29**, 1991, pp. 57–68.

18. C. A. Felippa, Parametrized variational principles for micropolar elasticity, *Internat. J. Solids Structures*, **29**, 1992, pp. 2709–2721.

19. C. A. Felippa and J. Schuler, Parametrized variational principles for electromagnetodynamics, in *Mechanics of Electromagnetic Materials and Structures*, ed. by J. S. Lee, G. A. Maugin and Y. Shindo, AMD-Vol 161, ASME, New York, 1993, pp. 171–194.

20. C. A. Felippa, B. Haugen and C. Militello, From the individual element test to finite element templates: evolution of the patch test, *Int. J. Numer. Meth. Engrg.*, in press.