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# AN ADAPTIVE TIME MARCHING STRATEGY FOR IMPES

**Stevens Paz[a], Alfredo Jaramillo[b], Rafael T. Guiraldello[b], Roberto F. Ausas[b], Fabricio S. Sousa[b], Felipe Pereira[c] and Gustavo C. Buscaglia[b]**

[a]*Departamento de Matemáticas, Universidad del Valle, Calle 100 #13-00 - stevens.paz@correounivalle.edu.co*

[b]*Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - ajaramil@icmc.usp.br, rafaeltrevisanuto@gmail.com, rfausas@icmc.usp.br, fsimeoni@icmc.usp.br, gustavo.buscaglia@icmc.usp.br*

[c]*Department of Mathematical Sciences, The University of Texas at Dallas - luisfelipe.pereira@utdallas.edu*

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**Abstract.** In a multiphase fluid system, the transport velocity can be related to the pressure through Darcy's law and it is coupled to a conservation law for the saturation variable of one of the phases. The resulting coupled system of elliptic and hyperbolic partial differential equations is used to the modeling of, for example, two-phase flows in oil reservoirs. The classical IMPES (IMplicit Pressure Explicit Saturation) method first solves the elliptic problem for the pressure and flux, and then updates the saturation with an explicit hyperbolic solver. This method is very costly, since the expensive elliptic solver must be invoked at time intervals defined by the stability limit of the hyperbolic solver. It is popular among users to update the flux every $C$ time steps, keeping it frozen in between, with $C$ determined by the user. In this work we propose a more accurate handling of the velocity and an automatic procedure for the selection of $C$ in IMPES codes. In the time steps at which the elliptic problem is not solved, the flux is extrapolated from previously computed values with polynomials of high degree. We also introduce an error estimator from which the correct value of $C$ can be derived without user intervention. The algorithm is very easy to implement. The results show that the proposed algorithm is stable, reliable and cost effective.

## 1 INTRODUCTION

Numerical schemes like IMPES (IMplicit Pressure Explicit Saturation) (Sheldon et al., 1959; Chen et al., 2004) and fully implicit methods (Collins et al., 1992; Yang et al., 2016) have been used in the industry to address multiphase problems like petroleum and gas production (Al-Hussainy et al., 1966; Greenkorn, 1983; Douglas et al., 1997), whose discretization results in large size linear systems (billions of unknowns) for the flow in heterogeneous domains. The IMPES scheme uses an operator splitting to deal with these large linear systems by separating the calculation of compositional variables (saturation) from the flow variables (pressure and velocity).

The explicit time marching step for saturation transport (hyperblic problem) is restricted by the CFL condition (Courant et al., 1928). So, when performing a typical simulation with the classical IMPES algorithm, in which the flow equations (Darcy's flow equations) solver is invoked as many times as the saturation transport solver, a very large number of time-expensive calls to the Darcy solver has to be done. One strategy to reduce the computational time spent in the simulation is to consider a larger time step (denoted $\Delta t_D$) for updating the flow variables (the computationally intensive part) than the time step for the transport variable (denoted $\Delta t_T$), complemented with a procedure to compute intermediate flux for time marching in the saturation transport.

As long as the data of successive velocity variables is available in an existing code, the last mentioned strategy is easy to implement, however, special attention is required for the selection of a **suitable** $\Delta t_D$ in terms of $\Delta t_T$. In some cases, $\Delta t_D$ is chosen based on physical intuition or computational limitations and the flow variables are frozen at their last computed values used as intermediate flux to feed the explicit scheme for saturations (see, e.g., (Douglas et al., 1997, 2000; Chen et al., 2004; Kou and Sun, 2010)).

In this article we present an adaptive time marching strategy for IMPES as a result of two main ideas. First, we use high order polynomial extrapolations for velocities in the IMPES algorithm aiming the calculation of more accurate intermediate saturations fields, instead of (locally) constant in time (zero-degree extrapolation) velocities. And second, we develop a criterion for choosing a good skipping number (the number of saturation transport steps between two consecutive velocity field updates), equivalently, for selecting $\Delta t_D$, inspired by adaptive integrators of ordinary differential equations. This strategy is applied to the layer 36 of the SPE10 benchmark.

## 2 THE TWO-PHASE MATHEMATICAL MODEL

The oil-water flow is modeled here by the Buckley-Leverett equation (Bear, 2013), which describes the transport of water saturation $s$, from now on just saturation, coupled with the Darcy's law, relating an incompressible Darcy velocity $\mathbf{u}$, or velocity for short, and the gradient pressure $\nabla p$. The system for this two-phase model within a heterogeneous domain $\Omega$ is

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = f(\mathbf{x}, t), \tag{1}$$

$$\mathbf{u}(\mathbf{x}, t) = -\lambda(s)\mathbf{K}(\mathbf{x})\nabla p(\mathbf{x}, t), \tag{2}$$

$$\phi \partial_t s(\mathbf{x}, t) + \nabla \cdot (\varphi(s)\mathbf{u}(\mathbf{x}, t)) = 0, \tag{3}$$

where $f$ is a source term in the mass conservation equation, $\mathbf{K}$ is the absolute permeability tensor, $\phi$ is the porosity of the medium (assumed constant), $\lambda$ is the total mobility and $\varphi$ is the

fractional function, these last given by (Corey and Brooks, 1975)

$$\lambda(s) = \frac{s^2}{\mu_w} + \frac{(1-s)^2}{\mu_o} \quad \text{and} \quad \varphi(s) = \frac{Ms^2}{Ms^2 + (1-s)^2}$$

expressed in terms of the oil and water viscosities, $\mu_o$ and $\mu_w$ respectively, and the quotient $M = \frac{\mu_o}{\mu_w}$. In this work, we will considered the time scaled by the constant porosity, for simplicity. The system (1)-(3), is completed with suitable initial and boundary conditions, such as imposed normal flow (or pressure) at $\partial\Omega$, initial saturation (Chen, 2001) and saturation values at inflows, dealing with configurations of injection and production wells approximated as volumetric sources. Regarding the existence and uniqueness of solutions of the two-phase problem, the reader is referred to the works of Chen (2001), Chen (2002), and Chen et al. (2006).

## 3 NUMERICAL SCHEME

Given a saturation field $s^n$ at a discrete time $t^n$, the classical IMPES strategy, to numerically solve the system (1)-(3), consist in the implicit calculation of the pressure $p^n$ and velocity $\mathbf{u}^n$ by means of the elliptic problem

$$\nabla \cdot \mathbf{u}^n = f(\mathbf{x}, t_n), \tag{4}$$

$$\mathbf{u}^n = -\lambda(s^n)\mathbf{K}(\mathbf{x})\nabla p^n, \tag{5}$$

which can be discretized using, e.g., cell-centered finite volumes. Thence, the saturation is explicitly updated to time $t^{n+1}$ by using a numerical scheme for the hyperbolic problem (3), namely

$$s^{n+1} = s^n + \Delta t^n \, G\left(s^n, \mathbf{u}^n\right), \tag{6}$$

where $G$ is a discrete transport operator such as the first order upwind scheme, for which the time step $\Delta t^n = t^{n+1} - t^n$ should be small enough so as to satisfy the CFL condition $\mathrm{CFL} = \frac{\Delta t^n \, \|\varphi'(s^n)\mathbf{u}^n\|_\infty}{h} \leq 1$, with $h$ being the spatial mesh size (Ewing, 1983; Douglas et al., 2000)

On a transient simulation, most of the computational time is spent in the calculation of pressure and velocity. In this way, strategies leading to less frequent velocity updates in the IMPES scheme are desirable, which induces one time step for the elliptic problem and a different time step for the hyperbolic problem (Chen et al., 2004).

### 3.1 The skipping number and saturation time marching

We denote by $t^n \in [0, T]$, $T > 0$, the dicrete time at which the Darcy's problem is solved and $\Delta t_D^n = t^{n+1} - t^n$ its $n$-th *coarse* time step. This time step is subdivided into $C^n$ substeps, leading to intermediate discrete times $t^{n,m} = t^n + \sum_{l=1}^{m} \Delta t_T^{n,l}$, $0 \leq m \leq C^n$, for the updating of transport saturation, so the *fine* time step $\Delta t_T^{n,m}$ must satisfy $\mathrm{CFL} \leq 1$. The integer number $C^n$, referred here as *skipping number*, determines the quantity of transport steps solved between two consecutive Darcy velocity calculation.

Intermediate saturation update, at time $t^{n,m}$, is computed using a velocity field $\mathbf{u}^{n,m}$, which is obtained by polynomial extrapolation of previously computed velocities $\mathbf{u}^n$, $\mathbf{u}^{n-1}$, etc., depending on the polynomial degree.

### 3.2 Adaptive Darcy time step

In principle, the skipping number $C^n$ can be chosen as an arbitrary constant, which, if too small, no time-computational gain is achieved and, if too large, possible important time variations in the flux will be ignored. An adaptive strategy to find *a good* $C^n$ (equivalently, to adapt

$\Delta t_D^n$) is then a pertinent feature to accelerate the IMPES scheme. We propose the following steps:

- Compute the velocities $\mathbf{u}^{n+1}$, coming from the Darcy's problem, and $\hat{\mathbf{u}}^{n+1}$, an extrapolated flux resulting from interpolating polynomials of degree $k \geq 0$. In particular, if $k = 1$, then

$$\hat{\mathbf{u}}^{n+1}(\mathbf{x}) = \frac{t^{n+1} - t^{n-1}}{\Delta t_D^{n-1}} \mathbf{u}^n(\mathbf{x}) - \frac{t^{n+1} - t^n}{\Delta t_D^{n-1}} \mathbf{u}^{n-1}(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

- Compute an error estimator $e = e(\mathbf{u}^{n+1}, \hat{\mathbf{u}}^{n+1})$. Supposing that this estimator has *a posteriori* polynomial error of the form $e \simeq \alpha (\Delta t_D^n)^{k+1}$, with $\alpha > 0$, and, if $\varepsilon \simeq \alpha (\Delta t_D^{n*})^{k+1}$ is a given tolerance for $e$ and $\Delta t_D^{n*}$ is the *ideal* time step, then

$$\frac{\varepsilon}{e} \simeq \left( \frac{\Delta t_D^{n*}}{\Delta t_D^n} \right)^{k+1}, \text{implying } \Delta t_D^{n*} \simeq \Delta t_D^n \left( \frac{\varepsilon}{e} \right)^{\frac{1}{k+1}},$$

  and the estimation of the skipping number is

$$C^{n*} = C^n \left( \frac{\varepsilon}{e} \right)^{\frac{1}{k+1}}.$$

- Check whether $e > \varepsilon$ or $e \leq \varepsilon$:

  (a) If $e > \varepsilon$ then redefine the previous constant $C^n$ of the interval $[t^n, t^{n+1}]$ as $C^{n,\text{new}} = \max\{C^{n*}, C_{\min}\}$ and go back to time $t^{n,1}$ and start over. Here, $C_{\min}$ is a minimum allowed value for the skipping number, for example, 1.

  (b) If $e \leq \varepsilon$ then define the skipping constant of the interval $[t^{n+1}, t^{n+2}]$ as $C^{n+1} = \min\left\{C^{n*}, \left\lfloor \frac{7}{4} C^n \right\rfloor\right\}$ and continue with the time marching The factor $\frac{7}{4}$ was empirically chosen.

  The substep (a) of going back to $t^{n,1}$, referred here as *backward approach*, is optional. Instead, before proceeding the time marching, one can use a *forward approach* by taking the obtained skipping number as the one to be used for the next time step $\Delta t_D^{n+1}$.

A convergence analysis of the extrapolation strategy for the IMPES algorithm is presented in Paz et al. (2021).

## 4 APPLICATION TO THE SPE10 BENCHMARK LAYER 36

The numerical scheme described in Section 3 requires the selection of the initial skipping number $C^0$, the degree of the polynomial $k$, the tolerance $\varepsilon$ for the error estimator and the error estimator function $e$, so as to perform the flux and saturation time marching. Cell-centered finite volumes were used to solve the elliptic and hyperbolic problems. The proposed adaptive time marching strategy for IMPES is tested here for the SPE10 benchmark layer 36, varying the previously mentioned parameters of the algorithm.

The permeability field shown in Figure 1 corresponds to the layer 36 of the SPE10 benchmark Christie et al. (2001) in a domain $\Omega = [0, 670.56\,\text{m}] \times [0, 365.76\,\text{m}]$, divided by a regular partition of $220 \times 60$ cells. For this simulation, two volumetric sources, $\Omega^{\text{L}}$ and $\Omega^{\text{R}}$, are located

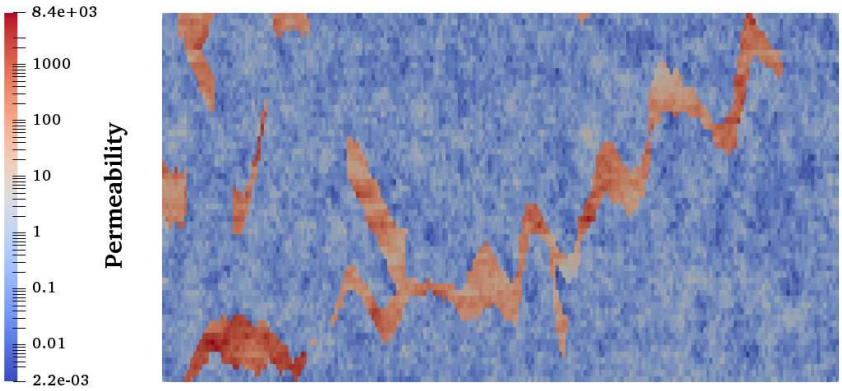


Figure 1: SPE10 layer 36 permeability field.

at the left and right boundaries and act as injector and production sources, respectively. So, when water saturation is injected in $\Omega^{\mathrm{L}}$, we have the oil produced fraction at $\Omega^{\mathrm{L}}$ given by

$$\mathcal{P}_{\mathrm{oil}}(t) = 1 - \frac{\int_{\partial\Omega^{\mathrm{R}}} \varphi(s(t))\mathbf{u}(\mathbf{x},t)\cdot d\mathbf{\Gamma}}{\int_{\partial\Omega^{\mathrm{R}}} \mathbf{u}\cdot d\mathbf{\Gamma}}. \tag{7}$$

Also, we impose no-flow boundary conditions, the initial condition is $s(\mathbf{x},0)=0$ for $\mathbf{x}\in\Omega\backslash\Omega^{\mathrm{L}}$ and, for time, we introduce the dimensionless time

$$T_{PVI}(t) = \frac{1}{V_p}\int_0^t \int_{\Omega_{\mathrm{w}}^1} f(\mathbf{x},\tau)\, d\mathbf{x}\, d\tau,$$

being $V_p$ the reservoir's total pore-volume.

To evaluate the effect of the adaptive time step technique proposed in Section 3.2, we start by using polynomials of degree $k=2$ to extrapolate the Darcy velocity at both fine and coarse time steps, and estimating the errors with

$$e = e\left(\mathbf{u}(\cdot,t),\hat{\mathbf{u}}(\cdot,t)\right) = \frac{\|\mathbf{u}(\cdot,t)-\hat{\mathbf{u}}(\cdot,t)\|_{L^2}}{\|\mathbf{u}(\cdot,t)\|_{L^2}}, \tag{8}$$

in simulations going beyond the breakthrough time with forward approach. As shown in Figure 2 (left), a restrictive tolerance for the estimator induces low values for the skipping number $C$. Before the breakthrough time, oscillations in the skipping number indicate important variations while estimating the Darcy flux, being necessary to reduce $C$ down to $5$ for $\varepsilon = 10^{-6}$, noticing that, in previous times, the skipping raised up to $41$.

Equation (7) defines the oil production curve which is also presented in Figure 2 (right) for different tolerances of the error estimator (8). The breakthrough time is well captured in all cases and no essential difference is found between the reference solution (Fine time, resulting from setting $C^n=1$) and the solutions calculated with the time adaptive strategy. The proposed algorithm allows for much more efficient computation without any appreciable loss of accuracy in the production curve, since the Darcy solver is invoked much less compared to the classical IMPES.

In order to assess accuracy gain with the adaptive strategy, we proceed to compare (a) the simulation with adapted $C^n$ and (b) another one in which the $C^n=43$ is fixed for all $n$ (corresponding to the average skipping number found in (a)) to the fine time solution resulting from

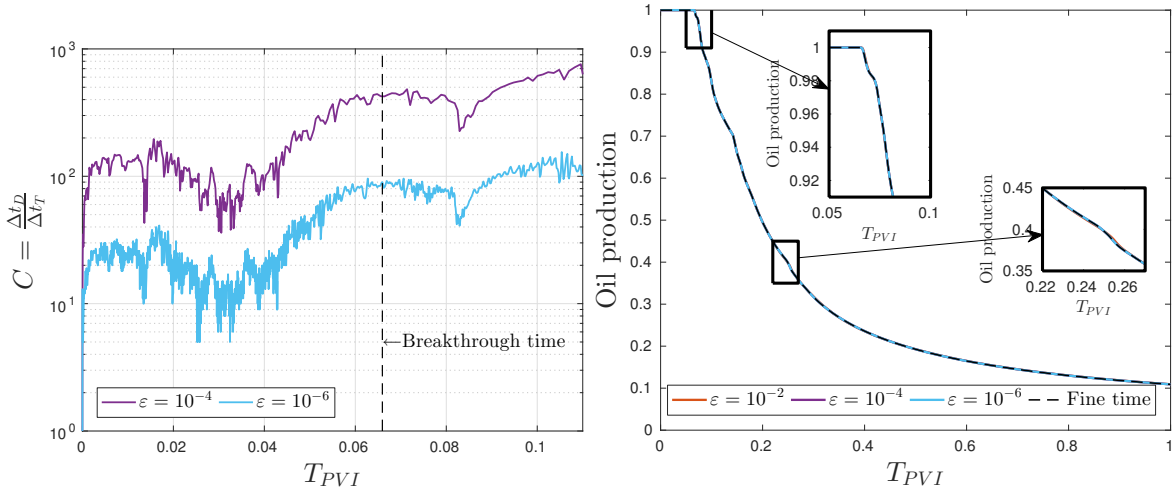


Figure 2: Skipping number as a function of time for $\varepsilon = 10^{-3}, 10^{-4}$ (left) and Oil production curve using three different values for the estimator tolerance $\varepsilon$ (right).

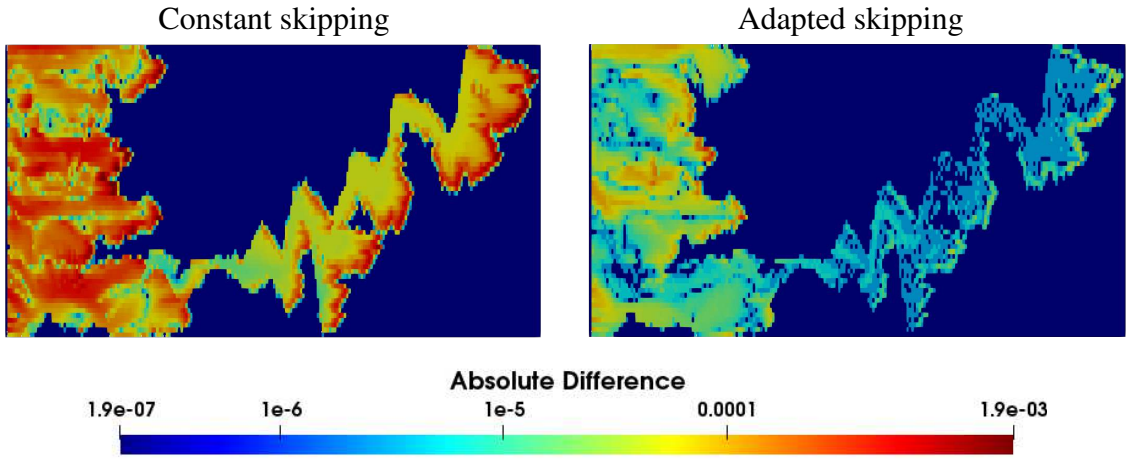


Figure 3: Absolute error between the saturation field computed using a constant skipping number $C^n = C^{\text{avg}}$ (left) and an adaptive time step (right) at time $T_{PVI} = 0.066$.

setting $C^n = 1$ for all time steps, using $k = 2$, $\varepsilon = 10^{-4}$ and forward approach. By design, simulations (a) and (b) take approximately the same computational time because they require to solve almost the same quantity of Darcy and transport problems. Figure 3 compares the spatial distribution of the saturation absolute error (based on the reference $C^n = 1$ solution) at time $T_{PVI} = 0.066$, showing that the adapted skipping number strategy locally recovers a better saturation field than the one coming from a constant skipping number strategy.

Finally, for simulations with adapted $C^n$ up to $T_{PVI} = 0.11$, Table 1 (left) contains the speedup, defined as the ratio of the computational time for $C^n = 1$ to that of the adaptive time step strategy, for different configurations resulting of varying the polynomial degree for extrapolation combined with the forward and backward approach. When $k = 0$, we already have a speedup of 7 (forward approach) and, for extrapolation of degree $k > 0$, speedups in the forward approach are almost the double of its backward counterpart. In terms of relative errors at time $T_{PVI} = 0.11$, also Table 1 shows no significant differences among the variants for

| $k$ | backward | forward |
|---|---|---|
| 0 | 6.4 | 7.2 |
| 1 | 14.8 | 25.8 |
| 2 | 13.7 | 28.7 |
| 3 | 11.5 | 24.7 |

| $k$ | backward | forward |
|---|---|---|
| 0 | $8.51 \cdot 10^{-5}$ | $8.45 \cdot 10^{-5}$ |
| 1 | $1.30 \cdot 10^{-5}$ | $4.49 \cdot 10^{-5}$ |
| 2 | $3.51 \cdot 10^{-5}$ | $7.38 \cdot 10^{-6}$ |
| 3 | $2.02 \cdot 10^{-5}$ | $2.11 \cdot 10^{-5}$ |

| $k$ | backward | forward |
|---|---|---|
| 0 | $7.74 \cdot 10^{-5}$ | $8.40 \cdot 10^{-5}$ |
| 1 | $1.22 \cdot 10^{-5}$ | $1.92 \cdot 10^{-5}$ |
| 2 | $1.84 \cdot 10^{-5}$ | $1.26 \cdot 10^{-5}$ |
| 3 | $2.13 \cdot 10^{-5}$ | $2.57 \cdot 10^{-5}$ |

Table 1: Speedup for different configurations (left), $L^2(\Omega)$ relative errors for velocity (center) and $L^1(\Omega)$ relative errors saturation (right).

both velocity and saturation fields compared to the fine time solution. So, regarding speedup and error control, the configuration $(k = 2, \text{forward})$ produced the best cost-benefit ratio for the simulation.

## 5 CONCLUSIONS

In this work we have modified the classical IMPES algorithm in such a way that the quantity of Darcy problems in a simulation is reduced and controlled by a certain tolerance and an error estimator that feed an adaptive time step process. The intermediate saturations are computed based on a polynomial extrapolations of previously calculated fluxes.

In the application to the layer 36 of the SPE10 benchmark, the use of quadratic polynomials with forward strategy gave us the best cost-benefit ratio in terms of accuracy and speedup. Also, for two different simulations performing approximately the same quantity of Darcy and transport problems (equivalently, spending the same computational time), the adaptive choice of $\Delta t_D$ produced more accurate solutions than the standard strategy of using a constant value of $\Delta t_D$.

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