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A GEOMETRIZATION OF THE CONFIGURATIONAL PROBLEM. PLASTICITY AND MICROMECHANICS OF NONSATURATED POROUS MEDIA

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Abstract. The primary objective of this work is to evaluate configurational forces by leveraging the geometric structure of the material manifold and the projection of the actual deformation onto it. Two classes of problems are addressed: finite-strain elastoplasticity and pseudo-finite-strain micromechanics in non-saturated porous media. General conditions for the strain energy function are established, and a global balance of pseudomomentum is formulated within a fully material manifold described in terms of a K-configuration. A natural frame of reference is introduced, characterised by K-parallelism (in the sense of Cartan), whereby each point of the elastic solid is associated with a unique stress-free reference configuration via the inverse of K. A Hencky-type form of the strain energy in the relaxed configuration is proposed, derived from general energetic and geometrical considerations. The resulting configurational forces are examined for two scenarios: (i) homogeneous elastoplastic solids with an arbitrary K-reference, and (ii) inhomogeneous porous media in the context of Biot-type formulations. A nonlinear expression for the configurational forces is obtained, which exhibits consistent convergence to the classical infinitesimal-strain formulation in the Biot case. The proposed framework provides a unified geometric basis for both elastoplasticity and micromechanics, and can be extended to other inhomogeneity-driven problems This behavior was numerically observed through the implementation of the boundary value problem using the Finite Element Method

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1 INTRODUCTION

Eshelbian mechanics or configurational mechanics may be regarded as a subset of the theory of material inhomogeneities, namely, when certain material properties such as density and elasticity coefficient undergo continuum variations even without external loadings. It is a discipline that mainly addresses a special type of force, called configurational force (in contrast to physical forces that are the structural response to an actual displacement of a material particle), that allows these inhomogeneities to be handled as defects such as inclusions, dislocations, fractures or, more generally, a sudden change at a certain material point without, as mentioned, external load (surface tractions, mass force, etc.).

The birth of a true configurational mechanic (Eshelbian mechanics) stems from Eshelby (1951) fundamental work as well as Kröner and Datta (1966) work. The Eshelbian mechanics tenet hinges on two concepts, the abovementioned configurational forces along with the Eshelby/Maxwell stress tensor (Eshelby, 1951). Initially connected to the field of material uniformity, Maugin (1993) revisited the fundamental connections between the Maxwell stress tensor and the variational principles thereby fostering the mathematical formulation of many other different fields.

With respect to geometrical aspects, in classical continuum mechanics, deformations are described within the Euclidean space $E^3 \simeq \mathbb{R}^3$. This framework is sufficient for regular displacement fields, yet it breaks down in the presence of defects. The very existence of such defects undermines the adequacy of modelling the material manifold as a simple Euclidean space. This recognition, developed during the second half of the twentieth century, motivated more sophisticated geometric backgrounds – non-Euclidean or even non-Riemannian – capable of encoding curvature and torsion. Such approaches, pioneered by Kondo (1955); Kröner (1958); Bilby et al. (1955); Stojanovic (1969); Noll (1967); Wang (1967), established a genuine geometrisation of continuum mechanics, conceptually akin to modern gravitation theories. Two key outcomes of this line of thought are: (i) its deep relation to the multiplicative decomposition of finite strains, and (ii) its ability to unify diverse theories such as volumetric growth and phase transitions through the notion of evolving local reference configurations.

Related to Biot theory for poroelastic bodies (Biot, 1941), two-phase and three-phase nonsaturated cases were investigated in a continuous porous media theory and by Lewis and Schrefler (1998). Mroginski et al. (2010) described an odd relationship between the vertical displacement and the degree of pollutant saturation. The environmental geo-mechanics problem was addressed by Schrefler (2001). Beneyto et al. (2015) presented a different approach for this issue based on the stress state decomposition technique (SSDT), and in (Di Rado et al., 2020)·, this same technique was extended to biological fields.

2 GEOMETRIZATION

2.1 Historical Background and Geometrical Foundations

As mentioned before, in real materials–such as metals–the presence of defects like dislocations, disclinations, or micro-inclusions challenges the idea of a Euclidean description. Dislocations, for instance, introduce displacement discontinuities quantified by the Burgers vector (Burgers, 1939), while disclinations reflect rotational incompatibilities. These irregularities call into question the conventional assumption of a globally Euclidean material manifold.

The mathematical formalization of such irregularities requires a departure from the Euclidean picture. To compare vectors, strains, or distortions at distinct material points in the presence of defects, one must specify how parallelism is defined across the body. This brings the problem naturally into the realm of differential geometry and, in particular, Cartan?s distinction between different notions of parallelism. With this in mind, (Cartan, 1931) established a distinction between different kinds of vector parallelism within the context in comparison with of classical Riemannian geometry. His approach rests on the idea that comparing vector magnitudes at separate points requires the introduction of a connection–that is, a path linking the entities to be compared, together with a derivative operator (a notion of connection that will later be generalized) regardless the metric tensor. Then, two principal notions of parallelism must be delineated:

a) Relative parallelism, which underpins conventional formulations in gravitational theory, is associated with the Levi-Civita connection–symmetric and torsion-free. In this framework, parallel transport is path-dependent, and vectors located at distinct points cannot be directly compared without specifying both a connecting curve and a connection along it.

b) Absolute or distant parallelism (géométrie à connexion complète) refers to a setting in which vectors at different points may be compared independently of the path connecting them. The connection may be flat (i.e., curvature-free), although torsion may be present–or even required. Specifically, for a flat connection with torsion, (Weitzenböck, 1923) introduced the concept of distant parallelism.

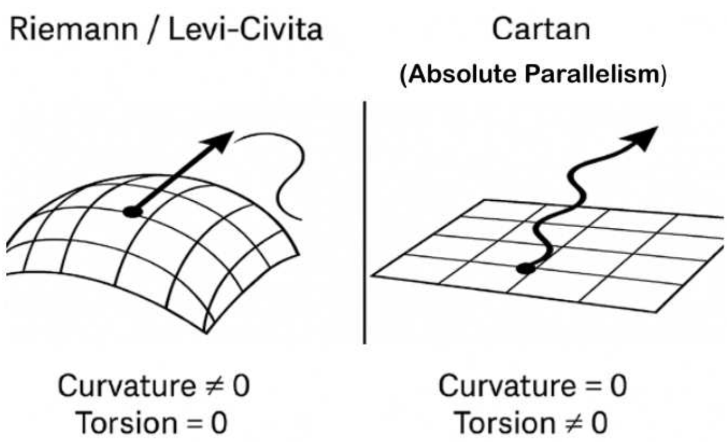

Figure 1: Different Notion of parallelism

An alternative geometrical interpretation of Cartan parallelism arises from a conceptual construction: assigning an arbitrary orthonormal frame to each point in space, and defining two vectors (located at points A and B) to be parallel if they exhibit identical components with respect to their respective local frames. In such a scenario, the reference frames themselves are also regarded as being in parallel.

Specifically, within the context of the present problem, torsion is the key geometric concept, originally formulated by Cartan and later incorporated into the theory of dislocations by Kondo (1955) and Kröner (1958). It quantifies the non-closure of infinitesimal parallelograms under parallel transport. From a physical standpoint, torsion serves as the natural geometric descriptor of dislocations, whereas curvature corresponds to disclinations. Importantly, this generalised, curvature-free manifold differs from standard Euclidean space in one essential aspect: it allows for non-zero torsion even when the metric remains Euclidean. Indeed, Cartan suggested that torsion, by itself, could provide a geometric resolution to several physical problems

This naturally raises the question: is it possible to formulate a material balance law grounded in geometric notions of parallelism, involving a covariant derivative reminiscent of those used in gravitational theory, but with structurally different properties?

An affirmative answer is given in (Maugin, 1993). By reformulating the strain energy function with reference to a position-dependent (or possibly fixed) local frame –allowing the energy at each point to be mapped onto its corresponding stress-free configuration (modulo a rigid-body rotation) –one achieves a geometric relaxation of the material body. This relaxed state is characterized by a linear transformation denoted K^{-1}. It is through this transformation that Cartan?s concept of K-parallelism is introduced into the mechanical framework, as will be discussed in the next section. Consequently, the energy function must be reformulated in terms of this linear mapping.

As a preview, dislocations –under various geometric interpretations– can be modeled by a flat connection with torsion, where the transformation K defines a structure of distant K-parallelism (the word distant stands for absolute parallelism without curvature). This leads naturally to a covariant derivative structure that governs both the material balance equations and the emergent configurational forces. The treatment of disclinations, which would require a curved connection, lies beyond the scope of the present discussion.

2.2 Energy strain function

In (Maugin, 1993), a detailed analysis of the formulation of the general features of the finite strain energy function is presented. Only the essential aspects will be recalled here. Let us assume the existence of a strain energy function per unit volume, depending on the strain gradient $\boldsymbol{F}$ in the local reference configuration and on the material point label $\boldsymbol{X}$.

$$W = W(\boldsymbol{F}, \boldsymbol{X}) \tag{1}$$

It may be postulated that each point of the elastic solid is associated with a unique stress-free reference configuration, denoted by K^{-1}. As previously noted, this transformation is a linear map that locally brings a neighborhood of X into a stress-free state. Under this assumption, the solid can be viewed as a collection of stress-free regions–referred to by Maugin (1993) as reference crystals–which, depending on the degree of K-parallelism between them, may not assemble into a globally compatible configuration. This idea is schematically illustrated in Fig. 2.

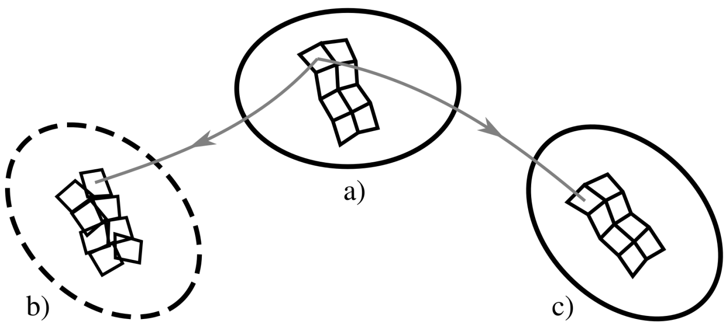

Figure 2: Several configurations: a) reference; b) crystal; c) arbitrary (e.g. current configuration)

The K^{-1} fields are not necessarily integrable over space (as illustrated in Fig. 2), since they do not arise from a global change of frame but are instead defined locally. However, in the case of a homogeneous body, and provided that the local torsion vanishes, it is possible to select the K tensors–via suitable additional rotations–such that all reference crystals are brought into ordinary parallelism (i.e., total parallelism without torsion, in Cartan?s terminology). Under this condition, the energy density W depends on the material point X only through K. In such cases, the K?1 fields become integrable, and the strain energy function can be explicitly written (Maugin, 1993).

$$\hat{W}(\boldsymbol{F},\boldsymbol{X}) = \boldsymbol{J}_K^{-1}\, W(\boldsymbol{F},\boldsymbol{K}(\boldsymbol{X})) \tag{2}$$

With $\boldsymbol{J}_K = \det(\boldsymbol{K})$. Geometrically, this corresponds to the case of a vector that, when parallel transported along a closed curve, preserves its direction and returns exactly to its initial position. This situation characterizes ordinary or total parallelism (i.e., parallelism without torsion) of the K tensors, which thus reflect the absence of geometric obstruction in the transport process.

In contrast, in the absence of material homogeneity, any form of dislocation –whether actual or fictitious– can be described geometrically by the parallel transport of a vector around a closed curve where the direction remains unchanged, yet the vector fails to return to its original position, ending instead in a displaced location. This situation must be mirrored in the definition of the $\boldsymbol{K}$ fields: the lack of total parallelism should be encoded in their geometric structure. This is the case associated with distant parallelism of the $\boldsymbol{K}$ fields. Consequently, when the body is not homogeneous, the $\boldsymbol{K}$ tensors cannot be integrated into a global configuration –that is, $\boldsymbol{K}$ is not the gradient of a smooth function and is therefore non-integrable over space, namely, while the any version of $\boldsymbol{F}$ is always the gradient of a deformation map (and thus integrable), $\boldsymbol{K}$ may not be. In this later case- -where $\boldsymbol{K}$ is not integrable–the strain energy function depends explicitly on $\boldsymbol{F}$, $\boldsymbol{K}$ and the material position $\boldsymbol{X}$ (Maugin, 1993).

$$\hat{W}(\boldsymbol{F},\boldsymbol{X}) = \boldsymbol{J}_K^{-1}\, W(\boldsymbol{F},\boldsymbol{K}(\boldsymbol{X})) = \bar{W}(\boldsymbol{F},\boldsymbol{K},\boldsymbol{X}) \tag{3}$$

which is tantamount to recognizing that the specific form of the strain energy depends on the chosen reference configuration.

2.3 Balance of pseudomomentum as a covariant derivative. Notion of K-connection.

Configurational forces assessment involves the explicit derivative of strain energy. Due to the K dependence, this derivative read:

$$\boldsymbol{f}^{\text{con}} = -\left.\frac{\partial \hat{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}} = -\frac{\partial \bar{W}}{\partial \boldsymbol{K}}\nabla_R \boldsymbol{K} - \left.\frac{\partial \bar{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}} \tag{4}$$

Using the relationship (Epstein and Maugin, 1990)

$$\frac{\partial \bar{W}}{\partial \boldsymbol{K}} = -\boldsymbol{b}\,\boldsymbol{K}^{-1} \tag{5}$$

and the balance of pseudomentum (Di Rado et al., 2025)

$$\boldsymbol{f}^{\text{con}} = -\nabla \boldsymbol{b} = -\left.\frac{\partial W}{\partial \boldsymbol{X}}\right|_{\text{exp}} \tag{6}$$

being $\boldsymbol{b}$ the Eshelby-Maxwell second order tensor $\boldsymbol{b} = W\,\boldsymbol{I} - \boldsymbol{F}^T\boldsymbol{P}$ (Eshelby, 1975), Eq. (4) leads to

$$\nabla \boldsymbol{b} = \underbrace{-\boldsymbol{b}\boldsymbol{K}^{-1}\nabla_R \boldsymbol{K}}_{(1)} + \underbrace{\left.\frac{\partial \bar{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}}}_{(2)} \tag{7}$$

The term labelled (1) involves a combination of the Eshelby tensor $\boldsymbol{b}$ and the K-maps, representing the contribution of inhomogeneities to the balance of pseudomomentum. It is clear that, due to the first term in equation (7), the dependence of the strain energy function on the material point $\boldsymbol{X}$ is twofold, thereby introducing a nonlinear contribution to the configurational force.

Term (2) in Eq. (7) corresponds to the classical effect of non-uniformity (i.e., material inhomogeneity), and it vanishes whenever the strain energy has no explicit dependence on position– namely, when the body is materially uniform.

A closer examination of term (1) enables a geometric interpretation. Assuming, for simplicity, that the body is homogeneous, Eq. (7) still holds and the geometric structure of the term becomes more transparent. Working on Eq. (7):

$$\nabla \boldsymbol{b} + \boldsymbol{b}\boldsymbol{K}^{-1}\nabla_R \boldsymbol{K} = 0 \tag{8}$$

Using a specially defined family of vector frame of reference $\vec{\Upsilon} = \boldsymbol{K}\frac{\vec{\partial}}{\partial \boldsymbol{X}}$, a K-parallelism can be introduced by requiring that two vectors at different points share the same components with respect to this frame. This notion of K-parallelism is typically tested through the vanishing of the covariant derivative. Accordingly, computing the covariant derivative of a vector field $\vec{V}$ relative to the given frame $\vec{\Upsilon}$ (being the gradient covector) yields (Hashiguchi and Yamakawa, 2013)

$$\vec{V} \otimes \tilde{\nabla} = \left(\frac{\partial}{\partial X} V\right) \otimes \mathrm{d}\tilde{X} = \left(K^{-1}V^k\right)_{,i} \vec{\Upsilon} \otimes \mathrm{d}\tilde{X} = \left(K_k^{-1m} V^k\right)_{,i} K_m^j \frac{\vec{\partial}}{\partial X} \otimes \mathrm{d}\tilde{X} \tag{9}$$

$$V^j_{;i} = \left((K_k^{-1m})_{,i} V^k K_m^j + V^k_{,i} K_k^{-1m} K_m^j\right) \frac{\vec{\partial}}{\partial X} \otimes \mathrm{d}\tilde{X} \tag{10}$$

$$V^j_{;i} = \left(V^k (K_k^{-1m})_{,i} K_m^j + V^k_{,i} \delta^j_k\right) \frac{\vec{\partial}}{\partial X} \otimes \mathrm{d}\tilde{X} = \left(V^j_{,i} + V^k \Gamma^j_{ki}\right) \frac{\vec{\partial}}{\partial X} \otimes \mathrm{d}\tilde{X} \tag{11}$$

In Eq. (11), a connection compatible with the field K, was defined, referred to as the K-connection, $\Gamma^j_{ki} = +(K^{-1})^m_{k,i} K^j_m = -(K^{-1})^m_k K^j_{m,i}$ (Maugin, 1993). Geometrically, the K-fields establish a structure of distant K-parallelism for vectors $\vec{V}$, as encoded by the connection $\vec{\Upsilon}$. Equation (11) remains, in components

$$V^j_{;i} = V^j_{,i} + V^k \Gamma^j_{ki} \tag{12}$$

Using this results in Eq. (8), the covariant divergence (K-covariant) of Eshelby stress tensor may be introduced:

$$b^j_{i;j} = b^j_{i,j} - b^j_k (K_i^{-1m})_{,j} K^k_m = b^j_{i,j} - b^j_k \Gamma^k_{ij} \tag{13}$$

Equation (13) indicates that, in the case of material uniformity, the K-covariant divergence of the Eshelby stress tensor vanishes. When the body is non-uniform, however, Eq. (7) takes the following form:

$$b^j_{i;j} = b^j_{i,j} - b^j_k \Gamma^k_{ij} = \left.\frac{\partial \bar{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}} \tag{14}$$

Using this nomenclature, the configurational forces are:

$$f^{\text{con}}_i = b^j_k \Gamma^k_{ij} - \left.\frac{\partial \bar{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}} \tag{15}$$

Together with the endomorphism defined by K, the K-connection defines a bond-space exhibiting two geometrical properties of fundamental importance for the mathematical modelling of crystal defects: curvature and torsion. Curvature is associated with disclinations, while torsion corresponds to dislocations. Upon analysing the K-connection, the emergence of non-symmetry of the connection coefficients with respect to the lower indices arises naturally.

$$\Gamma^j_{ki} - \Gamma^j_{ik} = -(K^{-1})^m_k K^j_{m,i} + (K^{-1})^m_i K^j_{m,k} \neq 0 \tag{16}$$

As previously mentioned, this skew-symmetric component is referred to as torsion, and it is precisely what Cartan identified with absolute K-parallelism –that is, a flat (curvature-free) space endowed with torsion (Choquet-Bruhat, 1968)

$$\Gamma^j_{ki} = \Gamma^j_{ki} - \Gamma^j_{ik} \tag{17}$$

In (Maugin, 1993), scaling the Eshelby tensor (weighted Eshelby tensor) $\bar{\boldsymbol{b}} = \boldsymbol{J}_K \boldsymbol{b}$, Eq. (13) becomes:

$$\bar{b}^j_{i;j} = \bar{b}^j_k T^k_{ji} + \bar{b}^j_i T^k_{jk} \tag{18}$$

This implies that the K-covariant derivative of the weighted Eshelby stress tensor can be expressed solely in terms of torsion -thus linking the configurational balance to the torsional content of the connection. Moreover, the connection defined via K is, by construction, curvature-free (Cartan, 1931). When disclinations are present in the body, curvature becomes the relevant geometric feature of the connection (Anthony, 1970) however, such cases will not be addressed in the present work. The K-connection thus acts as the geometric operator encoding the distant parallelism structure induced by K.

2.4 Energy forms for original and relaxed configuration under K-connection. K-derivative of $\bar{W}$

In the K-framework, the strain energy function can be expressed either with respect to the original configuration or with respect to the relaxed configuration defined by K^{-1} In the relaxed configuration, the energy depends on the deformation gradient F through its composition with K, that is, via FK (see Eq. (3)). Under the adopted conventions, the strain energy per unit volume in the original configuration is:

$$\hat{W}(\boldsymbol{F}, \boldsymbol{X}) = \boldsymbol{P} : \boldsymbol{F} \tag{19}$$

Being $\boldsymbol{P}$ the first Piola Kirchhof tensor. Using the fact that $\boldsymbol{J}_K^{-1} W(\boldsymbol{FK}, \boldsymbol{X}) = \bar{W}(\boldsymbol{F}, \boldsymbol{K}, \boldsymbol{X})$, the K-derivative of is:

$$\frac{\partial \bar{W}}{\partial \boldsymbol{K}} = \frac{\partial J_K^{-1}}{\partial \boldsymbol{K}} W + J_K^{-1} \frac{\partial W}{\partial \boldsymbol{F}\boldsymbol{K}} \frac{\partial \boldsymbol{F}\boldsymbol{K}}{\partial \boldsymbol{K}} \tag{20}$$

Assuming that (identity)

$$\frac{\partial W}{\partial \boldsymbol{F}\boldsymbol{K}} = J_K \boldsymbol{P}\boldsymbol{K}^{-1} \tag{21}$$

then

$$\frac{\partial \bar{W}}{\partial \boldsymbol{K}} = -J_K^{-1} \boldsymbol{K}^{-T} W + J_K^{-1} J_K \boldsymbol{P}\boldsymbol{K}^{-1}\boldsymbol{F} \tag{22}$$

$$\frac{\partial \bar{W}}{\partial \boldsymbol{K}} = \left(-\hat{W}\boldsymbol{I} + \boldsymbol{F}^T\boldsymbol{P}\right) \boldsymbol{K}^{-T} = -\boldsymbol{b}\boldsymbol{K}^{-T} \tag{23}$$

Equation (23) is due to (Epstein and Maugin, 1990) and matches Eq. (5).

In order to proceed, it is necessary to identify a suitable candidate for the strain energy function that satisfies Eq. (5). Among the possible choices, a Hencky-type formulation provides a consistent and natural option. Accordingly, the following expression is proposed:

$$\bar{W}(\boldsymbol{F}, \boldsymbol{K}, \boldsymbol{X}) = -\mathrm{Tr}\left(\boldsymbol{F}^T\boldsymbol{P}(\log \boldsymbol{K} - \boldsymbol{I} - \log J_K)\right) \tag{24}$$

If the trace is removed, then

$$\bar{W}(\boldsymbol{F}, \boldsymbol{K}, \boldsymbol{X}) = -\boldsymbol{F}^T : \boldsymbol{P} \log \boldsymbol{K} + \boldsymbol{F}^T : \boldsymbol{P} + \boldsymbol{F}^T : \boldsymbol{P} \log J_K \tag{25}$$

In Eq. (25), the relaxed configuration is expressed as a logarithmic function of K, in agreement with the Hencky-type strain measure discussed above. The dependence is specific to K, as previously established, and is fully consistent with finite-strain kinematics, provided that the restriction imposed by Eq. (23) is satisfied. Using The Mandel Stress tensor $\boldsymbol{M} = \boldsymbol{F}^T\boldsymbol{P}$

$$\bar{W}(\boldsymbol{F}, \boldsymbol{K}, \boldsymbol{X}) = -\mathrm{Tr}\left(\boldsymbol{M}(\log \boldsymbol{K} - \boldsymbol{I} - \log J_K)\right) \tag{26}$$

3 GEOMETRIZATION OF ELASTOPLASTICITY

Maugin (1993) introduces two important definitions:

1. *We call pseudo-plastic effects in continuum mechanics those mechanical effects –due to any physical property– that manifest themselves just like plasticity, through the notion of eigenstrains and eigenstresses in the language of Kröner (1958).*

2. *We call pseudo-inhomogeneity effects in continuum mechanics those mechanical effects –of any origin– that manifest themselves as so-called material forces in the material (Eshelbian) mechanics of materials.*

From this perspective, elastoplasticity offers a direct means of geometrically representing inhomogeneity via a specific choice of the K-map, which in this case may be identified with the plastic part of the deformation gradient. Considering the multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}^e\boldsymbol{F}^p$ of the total deformation gradient, it effectively captures the irreversible part of the deformation associated with pseudo-inhomogeneity

$$\boldsymbol{F} = \boldsymbol{F}^e \boldsymbol{F}^p \tag{27}$$

$$\boldsymbol{K}^{-1} = \boldsymbol{F}^p \tag{28}$$

$$\boldsymbol{F}\boldsymbol{K} = \boldsymbol{F}(\boldsymbol{F}^p)^{-1} \tag{29}$$

Equation (27) shows that the selection of K corresponds to a local relaxation of the plastic phenomenon at each point –this being, in fact, the original motivation for introducing K as a moving reference. If α denotes an internal state variable associated with dissipation mechanisms (e.g., damage, inelasticity, etc.), then the strain energy function takes the form:

$$\hat{W}(\boldsymbol{F}, \alpha) = \bar{W}(\boldsymbol{F}, \boldsymbol{K}(\alpha)) \tag{30}$$

Identifying $\boldsymbol{K}$ with $(\boldsymbol{F}^p)^{-1}$ and due to the relationship between $\boldsymbol{F}^p$ and α, then $\alpha^{-1} \approx \boldsymbol{K}$. According to Eq. (5) and the definition of the thermodynamic force $\boldsymbol{A}$ (Maugin, 1993)

$$\boldsymbol{A} = \frac{\partial \hat{W}}{\partial \alpha} = -\left(\frac{\partial \bar{W}}{\partial \boldsymbol{K}}\right)^T \frac{\partial (\boldsymbol{K}^T)}{\partial \alpha} \tag{31}$$

$$\boldsymbol{A} = \boldsymbol{b}\boldsymbol{K}^{-T} \frac{\partial (\boldsymbol{K}^T)}{\partial \alpha} = \boldsymbol{b}(\boldsymbol{F}^p)^T \frac{\partial (\boldsymbol{K}^T)}{\partial \alpha} \tag{32}$$

Using Eq. (8) but recalling that elastoplasticity is a homogeneous phenomenon, yields the following expression for the configurational force

$$\boldsymbol{f}^{\text{con}} = \boldsymbol{b}\boldsymbol{K}^{-1} \nabla_R \boldsymbol{K} = -\boldsymbol{b}\boldsymbol{K} \nabla_R \boldsymbol{K}^{-1} = -\boldsymbol{b}(\boldsymbol{F}^p)^{-T} (\nabla_R \, \alpha) \tag{33}$$

Or, in terms of $\boldsymbol{A}$:

$$\boldsymbol{f}^{\text{con}} = \boldsymbol{b}\boldsymbol{K}^{-1} \nabla_R \boldsymbol{K} = \boldsymbol{b}\boldsymbol{K}^{-T} \frac{\partial (\boldsymbol{K}^T)}{\partial \alpha} (\nabla_R \, \alpha)^T = \boldsymbol{A} (\nabla_R \, \alpha)^T \tag{34}$$

Leading to an alternative expression of $\boldsymbol{A}$:

$$\boldsymbol{A} = -\boldsymbol{b}(\boldsymbol{F}^p)^{-T} \tag{35}$$

4 GEOMETRIZATION MICROMECHANICS OF NON-SATURATED POROUS MEDIA

Recalling Eq. (4), the configurational forces are:

$$\boldsymbol{f}^{\text{con}} = -\frac{\partial \bar{W}}{\partial \boldsymbol{K}} \nabla_R \boldsymbol{K} - \left.\frac{\partial \bar{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}} \tag{36}$$

Applying similar reasoning to that used in plasticity for Eqs. (27)-(29), the relaxed configuration for Biot's micromechanical problem can be found using a multiplicative decomposition. This requires adopting a finite-strain formulation for the inhomogeneity source, i.e., the equivalent strain (Alhasadi and Salvatore, 2017; Eshelby, 1975). However, the complete micromechanical formulation has traditionally been developed within the framework of infinitesimal strain (Dormieux et al., 2006), which remains the most natural and widely adopted approach (Di Rado et al., 2025):

$$\boldsymbol{\varepsilon}_T^{*\alpha} = \mathbb{S}^{-1} : (\bar{\varepsilon}^{\alpha} - \boldsymbol{\Xi}) \tag{37}$$

Consequently, geometrising the problem under a finite-strain perspective entails enforcing the following relationship:

$$\boldsymbol{F}_T^{*\alpha} = \boldsymbol{I} + \boldsymbol{\beta}_T^{*\alpha} =\approx \boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha} \tag{38}$$

Here, $\boldsymbol{F}_T^{*\alpha}$ represents a linearised finite strain approximation of the aforementioned inhomogeneity source. In this context –and based on the nature of the problem– rigid body rotations are neglected.

Since the overall aim is to move from the original reference configuration to a relaxed one, the most suitable choice for K is

$$\boldsymbol{F}_T^{*\alpha}\boldsymbol{K} = \boldsymbol{I} \Longrightarrow \boldsymbol{K} \approx (\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha})^{-1} \tag{39}$$

Using Eqs. (39) and (5) in the first term of Eq. (36), the nonlinear part of the configurational forces, named $\boldsymbol{f}^{\text{int}}$, may be derived:

$$\begin{aligned} \boldsymbol{f}^{\text{int}} &= \boldsymbol{b}\boldsymbol{K}^{-1}\nabla_R \boldsymbol{K} = -\boldsymbol{b}\boldsymbol{K}\frac{\partial(\boldsymbol{K}^{-1})}{\partial \boldsymbol{F}_T^{*\alpha}}\nabla \boldsymbol{F}_T^{*\alpha} \\ &= -\boldsymbol{b}\,(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha})^{-1}\,\nabla\,(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha}) = -\boldsymbol{b}\,(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha})^{-1}\,\nabla \boldsymbol{\varepsilon}_T^{*\alpha} \end{aligned} \tag{40}$$

This conclusion is fully consistent with the results obtained in the context of plasticity. Moreover, the thermodynamic force $\boldsymbol{A}$ is

$$\boldsymbol{A} = -\boldsymbol{b}\,(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha})^{-T} \tag{41}$$

In total correspondence with plasticity.

4.1 Derivation of the explicit energy expression in the relaxed configuration

For evaluating the explicit derivative of the strain energy in equation (32) with respect to the relaxed configuration in Biot?s problem, the following considerations apply. The interaction energy in the micromechanical framework is evaluated in the infinitesimal strain regime (as justified in Di Rado et al. (2025)) and is given by

$$\hat{W}(\boldsymbol{\varepsilon}, \boldsymbol{X}) = -\tfrac{1}{2}\boldsymbol{\Sigma} : \boldsymbol{\varepsilon}_T^{*\alpha} \tag{42}$$

The explicit derivative in Eq. (42) was evaluated by substituting the expression around the fictitious transformation strain and taking the limit with respect to a vanishing perturbation, carefully chosen for that purpose (Alhasadi and Salvatore, 2017).

$$\begin{aligned} -\frac{\partial}{\partial X}\int_D \hat{W}^{\top} &= -\frac{\partial}{\partial X}\left(-\tfrac{1}{2}\int_D \Sigma^s : \varepsilon_T^*\right) \\ &= -\tfrac{1}{2}\int_D \left(\Sigma^s(X) : \varepsilon_T^* - \nabla \otimes \Sigma^s(X) : \varepsilon_T^* - \Sigma^s(X) : \varepsilon_T^*\right) = \tfrac{1}{2}\int_D \nabla \otimes \Sigma^m : \varepsilon_T^* \end{aligned} \tag{43}$$

Using the outcomes without the integral form, leads to

$$-\left.\frac{\partial \hat{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}} = \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma} \tag{44}$$

The results ofEq. (44) may be applied in the present framework from two different standpoints:

4.1.1 Strictly geometrical convergence

Using the previous expression together with Eq. (25) –but adopting the interaction energy sign.

$$\bar{W}(\boldsymbol{F}, \boldsymbol{K}, \boldsymbol{X}) = \tfrac{1}{2}\text{Tr}\left(\boldsymbol{\varepsilon}_T^{*\alpha}\boldsymbol{\Sigma}(\log \boldsymbol{K} - \boldsymbol{I} - \log J_K)\right) \tag{45}$$

The previous and within the context of an infinitesimal approximation at the microscale, while still preserving the K-parallelism. Using the results of expression (44) , the explicit derivative of Eq. (45) can be expressed as

$$-\left.\frac{\partial \hat{W}}{\partial \boldsymbol{X}}\right|_{\text{exp}} = \tfrac{1}{2}\left(-\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma}\log \boldsymbol{K} + \boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma} + \boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma}\log J_K\right) \tag{46}$$

Equations (40) and (46) together represent the total configurational force expressed in Eq. (36). Then

$$\begin{aligned} \boldsymbol{f}^{\text{con}} = -\boldsymbol{b}\left(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha}\right)^{-T}\nabla\boldsymbol{\varepsilon}_T^{*\alpha} - \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma}\log\left(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha}\right)^{-1} \\ + \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma} + \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma}\log\left(\det(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha})^{-1}\right) \end{aligned} \tag{47}$$

When the effect of the K-parallelism is disregarded, then $\boldsymbol{K} = (\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha})^{-1} \approx \boldsymbol{I}$, turning Eq. (47) into

$$\begin{aligned} \boldsymbol{f}^{\text{con}} = -\boldsymbol{b}\boldsymbol{K}^{-T}\underbrace{\nabla\boldsymbol{K}^T}_{=0} - \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma}\underbrace{\log \boldsymbol{K}}_{=0} \\ + \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma} + \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma}\underbrace{\log(\det\ \boldsymbol{K})}_{=0} \end{aligned} \tag{48}$$

If the contribution associated with K-parallelism is neglected, Eq. (47) reduces to its purely linear term, recovering the expression previously obtained in (Di Rado et al., 2025). This corresponds to the case where configurational effects arise solely from the classical infinitesimal-strain formulation.

4.1.2 Strictly mathematical convergence

Following Alhasadi and Salvatore (2017) treatment in small strains, we adopt here an explicit-derivative rule whereby the material gradient acts on the stress field only, while the inhomogeneity descriptor is kept frozen as a geometric reference. The neglected remainder (K-derivative) vanishes under a frozen-K hypothesis (piecewise-constant K at the element level or scale separation l<<<L). Writing

$$\bar{W}(\boldsymbol{F},\boldsymbol{K},\boldsymbol{X})\big|_{\mathrm{K}} = \tfrac{1}{2}\mathrm{Tr}\left(\boldsymbol{M}(\log\boldsymbol{K} - \log J_K\boldsymbol{I})\right) \quad ; \quad \Phi(\boldsymbol{K}) = \log\boldsymbol{K} - \log J_K\boldsymbol{I} \tag{49}$$

Expression (49) still satisfies condition (23). In earlier versions of the relaxed energy, see Eq. (26), the constant term $-\boldsymbol{I}$ had been introduced as a gauge in order to recover the classical product $\boldsymbol{F}^T\boldsymbol{P}$ in the limit $\boldsymbol{K} = \boldsymbol{I}$. In the present formulation, however, this adjustment is no longer required, since the role previously played by $-\boldsymbol{I}$ is now consistently replaced by the Mandel linearisation:

$$\boldsymbol{M} = \boldsymbol{F}^T\boldsymbol{P} = \boldsymbol{C}.\boldsymbol{S} = \boldsymbol{S} + O(\varepsilon^2) \quad ; \quad \boldsymbol{\Sigma} = \boldsymbol{S} + O(\varepsilon^2) \Longrightarrow \boldsymbol{M} \approx \boldsymbol{\Sigma} \tag{50}$$

As a result, the gauge $-\boldsymbol{I}$ has no physical meaning at this stage and may be safely omitted, eliminating spurious contributions while leaving the configurational force unaffected. Thus, following Eq. (45), the derivative reads

$$-\frac{\partial\hat{W}}{\partial\boldsymbol{X}}\bigg|_{\mathrm{exp}} = \tfrac{1}{2}\Phi(\boldsymbol{K}) : \nabla\boldsymbol{\Sigma} \tag{51}$$

Equations (40) and (51) together represent the total configurational force expressed in Eq. (36). Then

$$\boldsymbol{f}^{\mathrm{con}} = -\boldsymbol{b}\left(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha}\right)^{-T}\nabla\left(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha}\right) + \tfrac{1}{2}\Phi(\boldsymbol{K}) : \nabla\boldsymbol{\Sigma} \tag{52}$$

When the effect of K-parallelism is disregarded, Eq. (52) must be projected onto the linear regime. The first term in Eq. (52) is of second order and can therefore be discarded. The linearization of the second term requires:

$$\log\boldsymbol{K} = \log\left(\boldsymbol{I} + \boldsymbol{\varepsilon}_T^{*\alpha}\right) \approx \boldsymbol{\varepsilon}_T^{*\alpha} - O(\varepsilon^2) \tag{53}$$

$$\log J = \mathrm{Tr}(\boldsymbol{\varepsilon}_T^{*\alpha}) - O(\varepsilon^2) \tag{54}$$

With these considerations, Eq. (52) becomes:

$$\boldsymbol{f}_{\mathrm{lin}}^{\mathrm{con}} = \tfrac{1}{2}\boldsymbol{\varepsilon}_T^{*\alpha} : \nabla\boldsymbol{\Sigma} - \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{\varepsilon}_T^{*\alpha}) : \nabla\boldsymbol{\Sigma} \tag{55}$$

The second term in Eq. (55) reduces to a combination of deviatoric and volumetric parts. In particular, a trace contribution emerges of the form $\mathrm{Tr}(\boldsymbol{\varepsilon}_T^{*\alpha}) : \nabla\mathrm{Tr}(\boldsymbol{\Sigma})$ which was not present in the purely infinitesimal formulation adopted in (Di Rado et al., 2025). The appearance of this volumetric term is not an inconsistency but rather a clarification: in the linear surrogate, the contribution was effectively gauged out, whereas in the present geometric formulation it arises naturally from the $\log J_K\boldsymbol{I}$ part of the relaxed energy.

4.1.3 Analysis of results

These two routes, Sections 4.1.1 and 4.1.2, are not contradictory but complementary. The Mandel?Cauchy projection highlights the full analytic lineage of the finite-strain energy, exposing the volumetric trace term that is physically meaningful in porous media. The Biot?switch-off projection, on the other hand, provides a straightforward connection to (Di Rado et al., 2025) infinitesimal formulation by construction. The presence or absence of the volumetric term therefore depends on the chosen projection: analytic (series) versus geometric (switch-off). Together, they clarify how the present K-framework both extends the classical configurational theory and rationalizes its reduction to the infinitesimal case.

4.2 Final remarks

It is worth noting that, despite the additional complexity introduced by the K-framework and the explicit expression of configurational forces in terms of both geometric and energetic contributions, the practical finite element implementation remains consistent with the classical route. In fact, the configurational force can still be evaluated through the divergence of the Eshelby tensor, exactly as in the infinitesimal-strain formulation (Di Rado et al., 2025). This apparent redundancy is, however, not a weakness but a strength of the present approach: it shows that the classical expression emerges as a particular limit of a broader geometric formulation. The role of the K-connection and the relaxed-configuration energy is to reveal the underlying structure of the problem –capturing the influence of torsion, non-integrable reference mappings, and finite inhomogeneities– while preserving compatibility with standard computational strategies. Thus, the present framework should be understood not as a replacement of the classical procedure, but as its geometrical extension, providing a consistent theoretical basis for future applications beyond the homogeneous or infinitesimal setting.

In this sense, one may envisage that, should a consistent finite-strain extension of Eshelby's problem be developed –together with a finite-strain micromechanical formulation of porous media– the divergence of the Eshelby stress itself would necessarily be modified by the nonlinear kinematics. Under such conditions, the explicit K-based formulation presented here would no longer appear as a mere reinterpretation, but as the natural framework in which configurational forces can be rigorously derived and implemented in the large-deformation regime. In this perspective, the present contribution anticipates a unified foundation for a future finite-strain Eshelbian micromechanics of porous media, where both the geometric and energetic terms explicitly influence the numerical treatment.

5 CONCLUSIONS

This work has presented a geometrical framework for the evaluation of configurational forces, based on the structure induced by the K-mapping and its associated K-connection. By introducing $\boldsymbol{K}^{-1}$ as a position-dependent linear transformation mapping each material point to a stress-free local reference, it becomes possible to reformulate the strain energy in a relaxed configuration. This formulation naturally incorporates Cartan?s concept of distant parallelism, allowing the explicit modeling of inhomogeneities and defects –particularly dislocations– through a ?at connection with torsion.

Two representative problems were considered: finite-strain elastoplasticity and the micromechanics of non-saturated porous media in the context of Biot-type theory. In both cases, the K-framework yields a nonlinear expression for configurational forces, which, in Biot-type theory, reduces smoothly to the classical infinitesimal-strain results in the appropriate limit. The Hencky-type logarithmic form adopted for the relaxed configuration energy was shown to be consistent with the geometric interpretation of $\boldsymbol{K}$ and with established mechanical principles.

The approach provides a direct geometrisation of inhomogeneity, linking the torsion of the K-connection to configurational force generation. This formulation can be extended to other classes of material defects, and future work will address the inclusion of curvature effects associated with disclinations, thus completing the geometric description.

Although the present formulation recovers the classical evaluation of configurational forces via the divergence of the Eshelby tensor, this consistency should not be seen as a limitation. Rather, it highlights that the proposed K-framework provides the natural geometric extension required once finite-strain generalizations of both Eshelby?s problem and the micromechanics of porous media are pursued. In such a setting, the divergence of the Eshelby stress itself will be affected by nonlinear kinematics, and the explicit geometric?energetic structure developed here will become essential for a fully consistent implementation.

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