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MOVING FINITE-ELEMENT MESH MODELLING FOR INDUCED-VIBRATION ESTIMATION OF MOVING VEHICLES ON INFINITE ELASTIC MEDIA

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Abstract. This paper describes the development of one-dimensional beam elements and three-dimensional finite elements with moving meshes for applications on vibration estimation of semi-infinite elastic homogeneous media due to moving loads, such as trains on rail tracks or vehicles on roads. A conventional finite-element strategy requires very large meshes to allow the estimation of induced vibration of a moving vehicle, because a large portion of the mesh is required to model the distance travelled by the vehicle during simulation, in addition to a domain required at both sides of the moving vehicle at start and finish time of the simulation. An alternative approach is the use of moving elements to ensure that the loads do not approach the boundaries of the model, determining a significant reduction of the mesh size. The moving mesh moves at the speed of the vehicle, maintaining the contact-points location in the moving reference frame. Using this strategy, a time-invariant model can be obtained for constant velocity load or vehicle analysis in the case of homogeneous media. Random process modelling of roughness of the rails, track or road allows the assessment of its effect on induced vibration of moving vehicles on infinite media. Different vehicle models can be connected the moving mesh model, including different number of wheel axes by defining nodes of the mesh bellow each wheel, making the formulation very practical. Some application examples of the modelling technique are presented.

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1 INTRODUCTION

To predict ground-borne vibration due to railway traffic or different moving vehicles on roads, accurate computational models are required. Vibrations are caused by several excitation mechanisms, such as moving contact, moving loads, wheel and track unevenness, among other reasons. Several authors have studied moving loads or masses on infinite elastic domains (Steele, 1967; Fryba,1999; Anderson et al., 2001). Analytical solutions for the dynamic response of an infinite beam resting on a viscoelastic foundation and subjected to arbitrary dynamic loads have been developed by Yu and Yuang, 2014 among other authors.

Since domains are infinite, a typical finite-element (FE) mesh models a portion of the domain and incorporates appropriate absorbing boundary conditions to try to emulate the behavior of an infinite domain with the finite domain. In the cases of moving loads, moving mass in contact with the domain, or moving vehicle on a road or rail, the estimation of induced vibration may require large domains so that the moving contact/load stays within the mesh limits for the simulation time at the assumed velocity of the contact. This could require large meshes and substantial computational effort. As an alternative, this paper explores the use of moving meshes as a computationally efficient method to handle this type of problems. The method requires that the modeled infinite domain is homogeneous and allows the consideration of representing moving loads, moving masses in contact with the elastic domain or vehicle models in contact with the elastic domain at a finite number of points that maintain relative distance (see Figure 1).

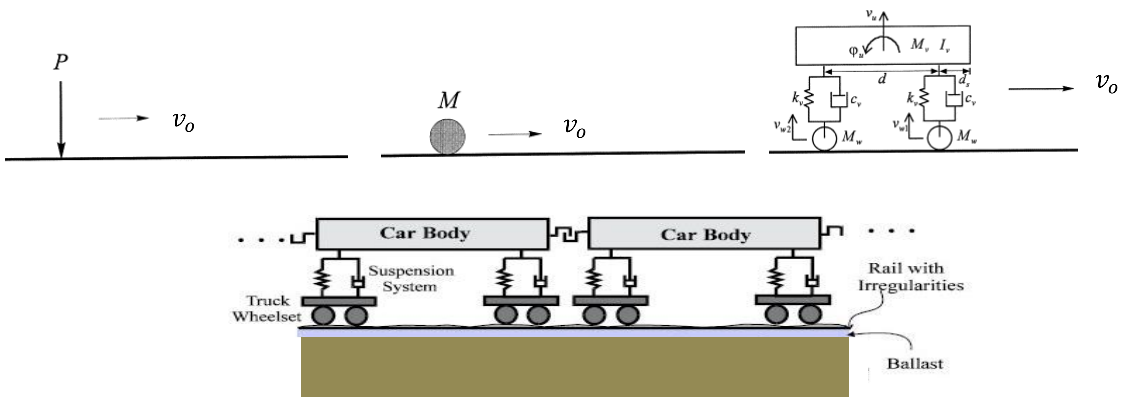

Figure 1. Problems of moving load, mass or vehicle on infinite elastic media.

The basic concept of moving mesh is to use a moving reference frame (relative coordinates), modelling the displacement fields using relative coordinates. The speed of the moving frame is that of the moving load/moving mass or moving vehicle.

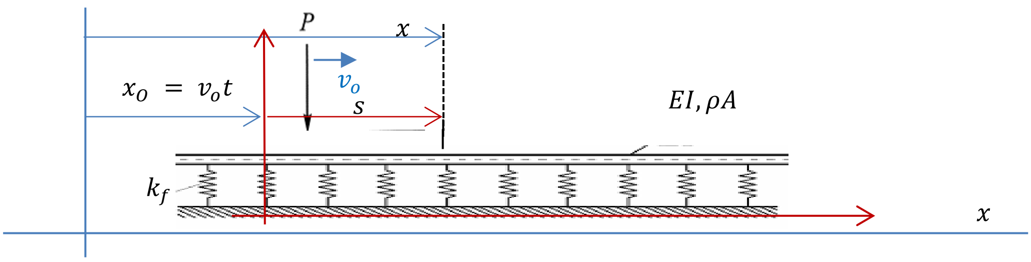

Figure 2. Moving load on a beam on elastic foundation.

To illustrate the concept of moving elements or moving frame of reference let's consider a

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homogeneous beam on elastic foundation supporting a moving load that moves at constant horizontal velocity v_o (Figure 2). The expressions of the time derivative of the vertical displacement field of a beam and its curvature are analyzed because these fields define the kinetic and elastic potential energy of a Bernoulli beam model. The vertical displacement $u(x,t)$ of the beam section located at a distance x from a fixed frame of reference (as that depicted in light blue in Figure 2) can be defined in a moving frame of reference at constant velocity v_o(as that depicted in red in Figure 2) using a relative coordinate s, such that $s = x - v_o\, t$, defining a function $r(s,t)$:

$$u(x,t) = r(s,t) = r(x - v_o\, t, t) \tag{1}$$

The vertical velocity of the section can be expressed in terms of the field $r(s,t)$ as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial r(s,t)}{\partial s}\frac{ds}{dt} + \frac{\partial r(s,t)}{\partial t} = -\frac{\partial r(s,t)}{\partial s} v_o + \frac{\partial r(s,t)}{\partial t} \tag{2}$$

since $\frac{ds}{dt} = -v_o$. On the other hand, because $\frac{ds}{dx} = 1$, the curvature of the Bernoulli beam for small deformations can be computed as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r(s,t)}{\partial s}\frac{ds}{dx}\right) = \frac{\partial}{\partial x}\left(\frac{\partial r(s,t)}{\partial s} 1\right) = \frac{\partial^2 r(s,t)}{\partial s^2}\frac{ds}{dx} = \frac{\partial^2 r(s,t)}{\partial s^2} \tag{3}$$

Equations (2) and (3) allow the construction of a discrete model of a moving mesh (moving reference frame) using standard interpolation functions, nodal displacements as generalized coordinates, and using a Lagrange formulation for the equations of motion in these generalized coordinates:

$$\frac{d}{d}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial U}{\partial \boldsymbol{q}} = \boldsymbol{Q_{nc}} \tag{4}$$

The kinetic energy of a single FE of length L (neglecting rotational inertia of the beam sections) can be expressed as

$$T = \frac{1}{2}\int_0^L \rho A \left(-\frac{\partial r(s,t)}{\partial s} v_o + \frac{\partial r(s,t)}{\partial t}\right)^T \left(-\frac{\partial r(s,t)}{\partial s} v_o + \frac{\partial r(s,t)}{\partial t}\right) ds \tag{5}$$

where ρ and A are the density and the cross section of the beam. Eq. (5) indicates that the kinetic energy of a continuum model discretized by interpolation functions in relative coordinates in a constant velocity reference frame, leads to an expression of kinetic energy with three terms: a quadratic form of time-derivatives of the nodal displacements, a linear form of the time-derivatives of the nodal displacements, and a quadratic for of the nodal displacements. This implies that additional terms in the differential equations in addition to mass matrix times the second derivatives of nodal displacements of conventional formulation with fixed reference frame will be present. On the other hand, because curvature (as Eq. (3) indicates) or strains in a general finite-element model with linear kinematics do not involve a change in the differential operator on $r(x,t)$ because $\frac{ds}{dx} = 1$, the elastic potential energy of the model using interpolating functions in terms of relative coordinates in a moving frame results in a quadratic form of the nodal displacements, leading to the same stiffness matrix for small nodal displacements that would result in a conventional finite-element with fixed reference frame.

This implies that a set of finite elements of homogeneous beam elements (Bernoulli or other models on elastic foundation) with moving reference frame can provide a very versatile method for estimating the motion of the beam with accuracy on the vicinity of the moving

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load/moving mass/moving vehicle in contact with the beam on elastic foundation. Because the elements located very far from the moving load/moving mass or moving vehicle in contact with the elastic supporting domain do not affect significantly the displacements in the vicinity of the moving load/moving mass or moving vehicle, a finite fixed-size mesh can be used to accurately estimate the displacements $r(s,t)$ using a discrete Lagrange formulation, that leads to differential equations in the nodal displacements/rotations that scale the interpolation functions in the assumed kinematic model.

To obtain the general form of the differential equations that derive from the type of kinetic energy that would imply the use of displacement fields interpolated as

$$r(s,t) = \sum_{i=1}^{N_q} N_i(r)\, q_i(t) \tag{5}$$

we use the Matlab® symbolic toolbox. The differential of the kinetic energy (dT) is obtained for a case of 2 generalized coordinates $q_i(t)$ ($i = 1{,}2$) and the Lagrange operator, $\frac{d}{dt}\frac{\partial dT}{\partial \dot{\boldsymbol{q}}} - \frac{\partial dT}{\partial \boldsymbol{q}}$, that leads to the differential equations contribution is applied to that kinetic energy differential, dT, to obtain the expressions to be integrated to compute the matrices that multiply $\ddot{q}(t)$, $\dot{q}(t)$, and $q(t)$ (M,G,H) in the differential equations of the model. The code and the analytical expressions of dM, dG and dH are shown for two generalized coordinates, $q_1(t)$ and $q_2(t)$. These expressions can be generalized for a general N-degree-of-freedom model.

```
syms q1 q2 q1d q2d q1dd q2dd vo rho N1 N2 N1p N2p N1pp N2pp
q=[q1;q2];       % Generalized coordinates
qd=[q1d;q2d];    % Generalized coordinates time derivatives
N=[N1 N2];       % Interpolation functions matrix
Np=[N1p N2p];    % Derivatives of interpolation functions matrix
Npp=[N1pp N2pp];% Second derivatives of interpolation functions matrix

ud=N*qd-vo*Np*q; % Velocity field ud
T=1/2*rho*transpose(ud)*ud; % Differential of kinetic energy
% First term of Lagrange operator on T
dTqd=[diff(T,q1d);diff(T,q2d)]; % First term of Lagrange operator on T
% Total time derivative of dTqd
d_dTqd_dt=diff(dTqd,q1)*q1d+diff(dTqd,q2)*q2d+diff(dTqd,q1d)*q1dd+diff(dTqd,q2d)*q2dd+
         +diff(dTqd,N1)*N1p*(-vo)+diff(dTqd,N2)*N2p*(-vo)+diff(dTqd,N1p)*N1pp*(vo)+ …
         diff(dTqd,N2p)*N2pp*(-vo);
% Second term of Lagrange operator on T
dT_dq=[diff(T,q1);diff(T,q2)];
% ODE differential equations obtained by Lagrange formulation
Lagrange_Eqs=d_dTqd_dt-dT_dq;
% Differential matrices of ODE in generalized coordinates q
dM(:,1)=diff(Lagrange_Eqs,q1dd);
dM(:,2)=diff(Lagrange_Eqs,q2dd);
dG(:,1)=diff(Lagrange_Eqs,q1d);
dG(:,2)=diff(Lagrange_Eqs,q2d);
dKvo(:,1)=diff(Lagrange_Eqs,q1);
dKvo(:,2)=diff(Lagrange_Eqs,q2);

% Computed analytical expressions of dM, dG and dH matrices are:
dM =[ N1^2*rho,   N1*N2*rho]
    [ N1*N2*rho,  N2^2*rho]

dG =[ -2*N1*N1p*rho*vo, -2*N1*N2p*rho*vo]
    [ -2*N2*N1p*rho*vo, -2*N2*N2p*rho*vo]

dH = [ N1*N1pp*rho*vo^2, N1*N2pp*rho*vo^2]
     [ N2*N1pp*rho*vo^2, N2*N2pp*rho*vo^2]
```

The integration of these differentials in the domain of an element gives the element matrices $\boldsymbol{M}_e$, $\boldsymbol{G}_e$, $\boldsymbol{H}_e$ of an arbitrary FE derived from kinetic energy for a set of interpolating

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functions. The inclusion of the elastic potential energy defines the element stiffness matrix $\boldsymbol{K}_e$ for the model in generalized coordinates. Therefore, the assembly at structural level of this type of element matrices in the absence of damping, leads to global matrices $\boldsymbol{M}, \boldsymbol{G}, \boldsymbol{H}$ and $\boldsymbol{K}$ that define the ordinary differential equations (ODE) of the form, where $\boldsymbol{F}(t)$ is an external loading vector (including gravity loads and other imposed generalized loads):

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{G}\dot{\boldsymbol{q}}(t) + (\boldsymbol{K} + \boldsymbol{H})\boldsymbol{q}(t) = \boldsymbol{F}(t) \tag{6}$$

In section 2 the Bernoulli FE on elastic foundation is presented for constant moving velocity v_o and some application examples of moving load, moving mass, and moving vehicle on an infinite homogeneous beam are developed. In section 3 a 3-D 20-node FE for homogeneous elastic properties is developed and application example is presented.

2 MOVING BEAM FINITE ELEMENT

In this section we derive the equations of motion of a Euler-Bernoulli beam on visco-elastic foundation using standard Hermite polynomials $N_1(s) = 1 - \frac{3}{L^2}s^2 + \frac{2}{L^3}s^3$; $N_2(s) = s - \frac{2}{L}s^2 + \frac{1}{L^2}s^3$; $N_3(s) = \frac{3}{L^2}s^2 - \frac{2}{L^3}s^3$; $N_4(s) = -\frac{1}{L}s^2 + \frac{1}{L^2}s^3$; as interpolation functions for a straight beam with 2 nodes with generalized coordinates given by transverse displacement of the left node, rotation of the left node, transverse displacement of the right node, and rotation of the right node:

$$r(s,t) = \boldsymbol{N}(s)\,\boldsymbol{q}(t) \qquad \boldsymbol{N} = [N_1(s)\;\; N_2(s)\;\; N_3(s)\;\; N_4(s)] \tag{7}$$

From Eq. (5) and Eq. (7) the kinetic energy of a single FE (neglecting rotational inertia of the beam sections) can be expressed as

$$T = \frac{1}{2}\int_0^L \rho A \left(-\frac{d\boldsymbol{N}}{ds}\,\boldsymbol{q}(t)\,v_o + \boldsymbol{N}(s)\dot{q}(t)\right)^T \left(-\frac{d\boldsymbol{N}}{ds}\,\boldsymbol{q}(t)\,v_o + \boldsymbol{N}(s)\dot{q}(t)\right) ds \tag{8}$$

The potential energy can be expressed as

$$U = \frac{1}{2}\int_0^L EI \left(\frac{d^2\boldsymbol{N}}{ds^2}\,\boldsymbol{q}(t)\right)^T \left(\frac{d^2\boldsymbol{N}}{ds^2}\,\boldsymbol{q}(t)\right) ds + \frac{1}{2}\int_0^L k_f\,(\boldsymbol{N}\,\boldsymbol{q}(t))(\boldsymbol{N}\,\boldsymbol{q}(t))^T ds + \frac{1}{2}\int_0^L g\rho\boldsymbol{N}\,\boldsymbol{q}(t) ds \tag{9}$$

where E, I, k_f and g are the Young modulus of the beam, the second moment of area of the beam, the elastic foundation parameter, and the acceleration of gravity, respectively.

Applying the Lagrange operator $\frac{d}{d}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial U}{\partial \boldsymbol{q}} = \boldsymbol{Q_{nc}}$ considering the non-conservative force projections $\boldsymbol{Q_{nc}}$ resulting from viscous foundation with parameter c_f and external loads the equations of motion can be obtained (using the symbolic toolbox of Matlab) for a specific element to yield:

$$\boldsymbol{M}_e\ddot{q}(t) + (\boldsymbol{G}_e(v_o) + \boldsymbol{C}_{ef})\dot{q}(t) + (\boldsymbol{K}_{eb} + \boldsymbol{K}_{ef} + \boldsymbol{H}_e(v_o^2) + \boldsymbol{K}_{cf}(v_o))q(t) = \boldsymbol{F}_{nc}(t) \tag{10}$$

The element matrices $\boldsymbol{M}_e$, $\boldsymbol{K}_{eb}$, $\boldsymbol{K}_{ef}$, $\boldsymbol{H}_e$, $\boldsymbol{G}_e$, $\boldsymbol{C}_{ef}$ and $\boldsymbol{K}_{cf}$ are computed using the Hermite polynomials using the symbolic toolbox. For the case of constant velocity of the reference frame and homogeneous mechanical properties of the beam and foundation, all matrices of the beam model are time invariant. This is a very convenient characteristic for analysis. H_e and G_e are non-symmetric matrices, and M_e, K_{eb} and K_{ef} are symmetric matrices. By a standard assembly procedure the time-invariant matrices of the full model (portion of the beam on elastic foundation) can be computed.

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```
Me=int(transpose(N)*rho*A*N,s,0,L); % Integration
Me=rho*A*L*[13/35, (11*L)/210,        9/70, -(13*L)/420;
           (11*L)/210,    L^2/105,  (13*L)/420,    -L^2/140;
           9/70, (13*L)/420,       13/35, -(11*L)/210;
           -(13*L)/420,   -L^2/140, -(11*L)/210,     L^2/105];

Keb=int(transpose(Npp)*E*A*Npp,s,0,L); % Integration
Keb=E*I/L^3*[ 12,   6*L,  -12,   6*L;
              6*L, 4*L^2, -6*L, 2*L^2;
             -12,  -6*L,   12,  -6*L;
              6*L,  2*L^2, -6*L, 4*L^2]

Kef=int(transpose(N)*kf*N,s,0,L);
Kef=L*kf*[13/35,       (11*L)/210,        9/70, -(13*L)/420;
          (11*L)/210,    L^2/105,  (13*L)/420,    -L^2/140;
           9/70,       (13*L)/420,       13/35, -(11*L)/210;
          -(13*L)/420,  -L^2/140,  -(11*L)/210,     L^2/105]

Ge=int(2*transpose(N)*rho*A*vo*Np,s,0,L)
Ge=rho*A*vo*[1/2,   -L/10,  -1/2,    L/10;
             L/10,       0, -L/10, L^2/60;
             1/2,     L/10,  -1/2,  -L/10;
            -L/10, -L^2/60,  L/10,      0]

He=int(transpose(N)*Np*rho*A*vo^2,s,0,L)
He=rho*A*vo^2/L*[-6/5,   -(11*L)/10,  6/5, -L/10;
                 -L/10, -(2*L^2)/15, L/10, L^2/30;
                  6/5,         L/10, -6/5,  (11*L)/10;
                 -L/10,       L^2/30, L/10, -(2*L^2)/15]

Cef=int(transpose(N)*N*cf,s,0,L)
Cef =L*cf*[ 13/35, (11*L)/210,      9/70, -13*L /420]
          [ 11*L/210,    L^2/105,  13*L/420,    -L^2/140]
          [      9/70,   13*L/420,     13/35,  -11*L/210]
          [-13*L/420,   -L^2/140, -11*L/210,     L^2/105]

Kcf=-int(transpose(N)*Np*cf*vo,s,0,L)
Kcf = cf*vo*[   1/2,   -L/10,  -1/2,   L/10]
            [  L/10,       0, -L/10, L^2/60]
            [   1/2,    L/10,  -1/2,  -L/10]
            [ -L/10, -L^2/60,  L/10,      0]
```

If a moving external vertical load is applied to a particular node of the moving mesh, it is directly assembled at the contact node in the corresponding generalized coordinate. If a moving mass m (at constant velocity v_o) in contact with the moving FE beam element is analyzed, the contribution to the equations of motion of the generalized coordinates can expressed in terms of the relative coordinate of the mass in the FE, s_m, and the constant velocity v_o computing the corresponding element matrices (M_{em}, G_{em}, H_{em}) for assembly in the model:

```
Mem= transpose(N(s_m))*m*N(s_m))
Gem=2*transpose(N(s_m))*m*vo*Np(s_m))
Hem= transpose(N(s_m))*Np(s_m)*m*vo^2
```

These element matrices are assembled in the structural model matrices to obtain the ODE on the model, including the moving mass and corresponding moving weight. If the moving mass is located on a node of the mesh, with local coordinate $s_m = 0$, these matrices for Hermite polynomial interpolation functions take simple expressions:

```
Mem = [ m, 0, 0, 0]        Gem = [ 0, 2*m*vo, 0, 0]       Hem = [ 0, m*vo^2, 0, 0]
      [ 0, 0, 0, 0]              [0,      0, 0, 0]              [ 0,       0, 0, 0]
      [ 0, 0, 0, 0]              [0,      0, 0, 0]              [ 0,       0, 0, 0]
      [ 0, 0, 0, 0]               [0,      0, 0, 0]             [ 0,       0, 0, 0]
```

Finally, if a vehicle model is moving at constant velocity v_o on the beam, the mass, damping, and stiffness matrices of the vehicle can be assembled at the structural level by

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extending the generalized coordinate vector of the beam with the generalized coordinates of the vehicle model and assembling the contributions of the mass, damping and stiffness matrices at the nodes of contact of the vehicle (wheels) on the beam. To simplify the assembly process, the beam mesh in relative coordinates must have nodes defined at the contact points of the vehicle wheels on the beam. In the following subsection some applications of moving load and moving vehicle are presented. It is worth mentioning that this type of models would lead to time-varying matrices in the ODEs in a formulation with not-moving mesh.

2.1 Application examples of beam-FE in moving coordinate system

Stationary deformation for constant moving load

A mesh of 250 elements and 251 nodes separated by FE of 1 m in length is created to analyze the stationary response of the beam on elastic foundation subjected to constant moving load (shown in Figure 3). The parameters considered for this example are $A = 1$; $\rho = 1$; $L = 1$; $E = 100000$; $v_o = 60$; $I = 1$; $P = 98.1$; $k_f = 100$, $c_f = 0$. Two stationary cases were considered: Case 1. Vertical external constant load P applied in the central node of the mesh (shown in light blue in Figure 3), and Case 2: Two vertical external loads of magnitude $P/2$ separated by 4 meters (as shown scaled in magenta in Figure 3). The right figure illustrates the estimation of stationary solutions of the vertical displacement field for these two cases, $r_{st}(s)$, computed with the model matrices assembled at a structural model level (251×2 dofs) solving for the particular time-independent solution $\boldsymbol{q}_p$ of the ODE system:

$$(\boldsymbol{K}_b + \boldsymbol{K}_f + \boldsymbol{K}_{cf} + \boldsymbol{H})\, \boldsymbol{q}_p = \boldsymbol{P} \tag{11}$$

As expected, minor differences are observed at sections far from the moving load and a smaller peak displacement under the applied loads is obtained for Case 2 (separated loads) with respect to Case 1 (concentrated load).

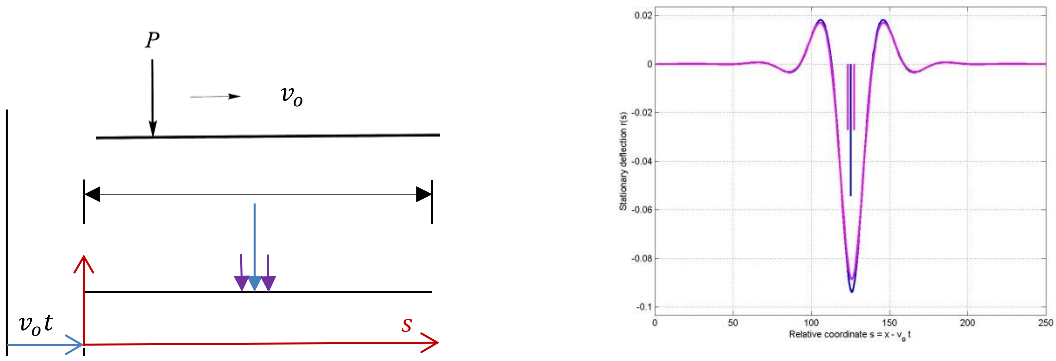

Figure 3. Stationary deformation field $r_{st}(s)$ for single vertical load (blue lines) and for the same vertical load applied in two nodes (magenta lines)

Stationary deformation for moving mass on beam on elastic foundation

To assess the difference in the particular solution between a moving load and moving mass modal, the contribution of the lumped mass to the particular solution equation including the moving mass is done using the same mesh defined in previous example for a single moving load and assembling the moving mass H_{em} element matrix in the global $\boldsymbol{H}_m$ matrix for the element whose left node corresponds to the location of the mass weight:

$$(\boldsymbol{K}_b + \boldsymbol{K}_f + \boldsymbol{K}_{cf} + \boldsymbol{H}_b + \boldsymbol{H}_m)\, \boldsymbol{q}_p = \boldsymbol{P} \tag{12}$$

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Figure 4 compares the stationary deformation pattern for moving load (previously computed) with that considering the moving mass (and weight). The mass is assumed as $m = P/g$ with $P = 98.1$ and the same parameters used in previous example. For the assumed parameters a very small difference is found in the stationary response of both models (moving load in blue line and moving mass and weight in red dots).

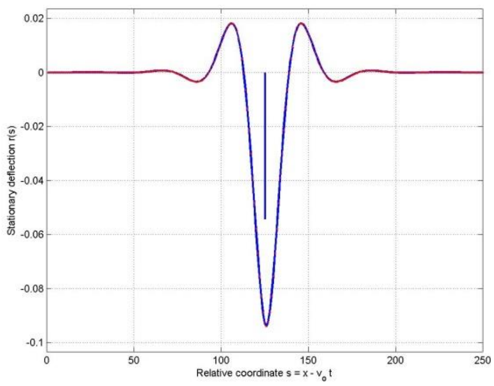

Figure 4. Stationary deformation field $r_{st}(s)$ for single vertical load (blue line) and for moving mass with the same vertical load (red dots).

Analysis of moving vehicle with roughness contact on a beam on elastic foundation

To illustrate the application of moving mesh to the analysis of a moving vehicle on a beam on elastic foundation, including roughness in contact, a simple model is developed with a spring k_n and no viscous damper in contact between vehicle and beam. Figure 5 shows the mechanical parameters of the model and the definition of the generalized displacements.

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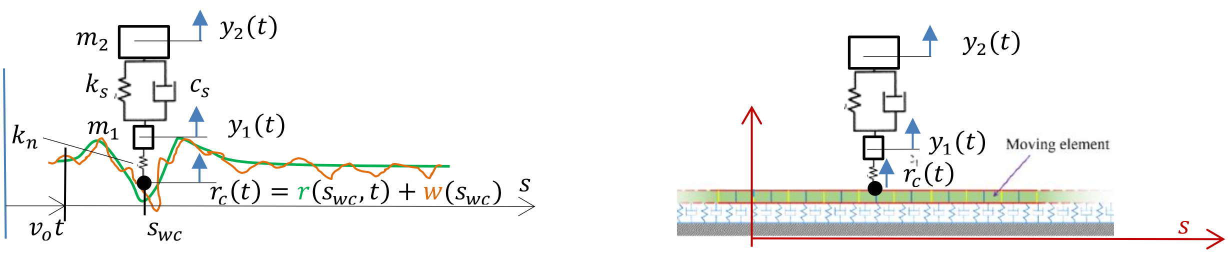

Figure 5. Quarter vehicle model on elastic media with roughness in contact.

If roughness of the wheel or road surface is to be included in to assess its influence in vibration of the road-vehicle system, the vertical displacement of the wheel contact can be expressed as:

$$r_c(t) = r(s_{wc}, t) + w(v_o t + s_{wc}) = L_{wc}\, \boldsymbol{q}(t) + w(v_o t + s_{wc}) \qquad (13)$$

where s_{WC} is the relative coordinate of the node of wheel contact under consideration, L_{wc} is the kinematic transformation from FE nodal displacements to $r(s_{wc}, t)$, and $w(v_o t + s_{wc})$ is the roughness vertical displacement model (deterministic or random).

To construct the model of the vehicle with the vertical displacements $r_c(t)$, $y_1(t)$, and $y_2(t)$ the mass, damping and stiffness matrices of the vehicle model (see Figure 4) are assembled:

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$$M_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \qquad K_v = \begin{bmatrix} k_n & -k_n & 0 \\ -k_n & k_n + k_s & -k_s \\ 0 & -k_s & k_s \end{bmatrix} \qquad C_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_s & -c_s \\ 0 & -c_s & c_s \end{bmatrix} \tag{14}$$

If we define as K_b the stiffness matrix of the beam on elastic foundation in coordinates $\boldsymbol{q}$ of the beam, and we partition the stiffness matrix of the vehicle model with the slave contact displacement $r_c(t)$ and the rest of the generalized coordinates of the vehicle model, $\boldsymbol{y}(t)$:

$$\boldsymbol{K_v} = \begin{bmatrix} K_{r_c r_c} & K_{r_c y} \\ {K_{r_c y}}^T & K_{yy} \end{bmatrix} = \left[\begin{array}{c|cc} k_n & -k_n & 0 \\ \hline -k_n & k_n + k_s & -k_s \\ 0 & -k_s & k_s \end{array}\right] \tag{15}$$

The elastic potential energy of the model can be expressed as:

$$U = \frac{1}{2} \begin{bmatrix} q \\ r_c \\ y \end{bmatrix}^T \begin{bmatrix} K_b & O & O \\ O & K_{r_c r_c} & K_{r_c y} \\ O & {K_{r_c y}}^T & K_{yy} \end{bmatrix} \begin{bmatrix} q \\ r_c \\ y \end{bmatrix} \tag{16}$$

Using the following kinematic transformation

$$\begin{bmatrix} q \\ r_c \\ y \end{bmatrix} = \begin{bmatrix} I & O \\ L_{wc} & O \\ O & I \end{bmatrix} \begin{bmatrix} q \\ y \end{bmatrix} + \begin{bmatrix} O \\ 1 \\ 0 \end{bmatrix} w(v_o t + s_{wc}) \tag{17}$$

The elastic potential energy can be reformulated using Eqs. (16) and (17):

$$U_e = \frac{1}{2} \left(\begin{bmatrix} I & O \\ L_{wc} & O \\ O & I \end{bmatrix} \begin{bmatrix} q \\ y \end{bmatrix} + \begin{bmatrix} O \\ 1 \\ 0 \end{bmatrix} w(v_o t + s_{wc}) \right)^T \begin{bmatrix} K_b + K_f & O & O \\ O & K_{r_c r_c} & K_{r_c y} \\ O & {K_{r_c y}}^T & K_{yy} \end{bmatrix} \left(\begin{bmatrix} I & O \\ L_{wc} & O \\ O & I \end{bmatrix} \begin{bmatrix} q \\ y \end{bmatrix} + \begin{bmatrix} O \\ 1 \\ 0 \end{bmatrix} w(v_o t + s_{wc}) \right) \tag{18}$$

Simplifyin, this can be written as

$$U_e = \frac{1}{2} \begin{bmatrix} q \\ y \end{bmatrix}^T \begin{bmatrix} K_b + K_f + {L_{wc}}^T K_{r_c r_c} L_{wc} & K_{r_c y} \\ {K_{r_c y}}^T & K_{yy} \end{bmatrix} \begin{bmatrix} q \\ y \end{bmatrix} + \\ + \frac{1}{2} K_{r_c r_c} w(v_o t + s_{wc})^2 + \\ + \begin{bmatrix} q \\ y \end{bmatrix}^T \begin{bmatrix} {L_{wc}}^T K_{r_c r_c} \\ {K_{r_c y}}^T \end{bmatrix} w(v_o t + s_{wc}) \tag{19}$$

Finally, applying $\frac{\partial U_e}{\partial q}$ and $\frac{\partial U_e}{\partial y}$ and considering the global road-vehicle model can be expressed as:

$$\begin{bmatrix} M_b & O \\ O & M_{yy} \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} C_f + G_b & O \\ O & C_{yy} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} H_b + K_b + K_f + K_{cf} + {L_{wc}}^T K_{r_c r_c} L_{wc} & K_{r_c y} \\ {K_{r_c y}}^T & K_{yy} \end{bmatrix} \begin{bmatrix} q \\ y \end{bmatrix} = \\ \begin{bmatrix} -M_b 1_z \\ -M_{yy} 1_z \end{bmatrix} g - \begin{bmatrix} {L_{wc}}^T K_{r_c r_c} \\ {K_{r_c y}}^T \end{bmatrix} w(v_o t + s_{wc}) \tag{20}$$

These differential equations allow the estimation of induced vibrations due to vehicle motion and due to roughness at constant circulation velocity. The stationary random process contribution of a stationary random roughness process to a specific output vector of interest

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$z(t)$ (such as nodal displacements or vehicle accelerations) can be characterized by the power spectral density matrix:

$$S_{zz}(\varpi) = H^{*}_{zw}(\varpi) S_{ww}(\varpi) H^{T}_{zw}(\varpi) \tag{21}$$

where $S_{ww}(\varpi)$ is the PSD of the road roughness modelled as a stationary random process and $H_{zw}(\varpi)$ is the frequency response function (FRF) from $w(t)$ to output vector $z(t)$ that can be computed by transforming to the frequency domain the ODE defined in Eq. (20). For example the FRF from $w(t)$ to generalized coordinates $z(t) = \begin{bmatrix} q \\ y \end{bmatrix}$ of the model can be computed solving for each frequency ϖ the linear set of equations:

$$\left(-\varpi^2 \begin{bmatrix} M_b & O \\ O & M_{yy} \end{bmatrix} + j\varpi \begin{bmatrix} C_b + G_b & O \\ O & C_{yy} \end{bmatrix} + \begin{bmatrix} H_b + K_b + K_f + K_{cf} + {L_{wc}}^T K_{r_c r_c} L_{wc} & K_{r_c y} \\ {K_{r_c y}}^T & K_{yy} \end{bmatrix} \right) H_{zw}(\varpi) = - \begin{bmatrix} {L_{wc}}^T K_{r_c r_c} \\ {K_{r_c y}}^T \end{bmatrix} \tag{22}$$

Eqs. (21) and (22) can be used to estimate the stationary random response of the model in the frequency domain. Alternatively, if a filtered white-noise random process model of the roughness process $w(t)$ is used, the stationary covariance matrix of the generalized coordinates can be computed using the Lyapunov equation formulating the model in an extended state space.

3 MOVING 3D 20-NODE FINITE ELEMENT

Applying the general expressions derived in Section 1 based on the Lagrange operator on the kinetic energy of finite elements in relative coordinates (moving mesh at constant speed) we derive the FE matrices M,G,H for a 20-node 3-dimensional parallelepiped FE assuming moving velocity in the X-direction. The symbolic expressions of the 60x60 matrices are obtained and coded for numerical evaluation of model matrices of moving elastic media, that can be used to represent supporting soil media of rails or roads. The main part of the code used to obtain the FE model matrices is shown:

```
%% Main steps of code for symbolic matrices computation of 20-Node FE
syms t r s E nu dx dy dz rho vo real;  % vo is assumed in direction X
XYZ_Elem=[XYZ(:,1)*dx XYZ(:,2)*dy XYZ(:,3)*dz];
N=funcionesForma2(t,r,s); % Create interpolation functions for 20-node isoparametric FE
[x,y,z]=coordenadasElementales3(XYZ_Elem,N);% Relative coordinates interpolation
[jacTranspuesta]=matrizJacobiana2(x,y,z,t,r,s); % Compute Jacobian matrix

%% Stiffness matrix computation
D=matrizD(E,nu); % Constitutive matrix
derivadaFormaxyz=derivadaFuncionesForma(N,r,t,s,jacTranspuesta);
B=matrizB(derivadaFormaxyz,N); % B matrix relates strains = B*u
dKelem=matrizRigidezParaIntegrar2(jacTranspuesta,B,D);% dKelem
Ke=integrateMatriz_Sym(dKelem,t,r,s);% Stiffness element matrix Ke

%% Computation of G and H matrices assuming vo in X-direction (t isoparametric coordinate)
% derivadaFormatrs=[diff(N,t);diff(N,r);diff(N,s)];
derivadaForma_t=[diff(N,t);zeros(1,20);zeros(1,20)];          % Only dN/dt (not dN/d or dN/ds)
derivadaForma_tt=[diff(diff(N,t),t);zeros(1,20);zeros(1,20)]; % Only d^2N/dt^2
for k=1:length(N); % 20 interpolation functions
        derivadaForma_x(:,k)=jacTranspuesta\derivadaForma_t(:,k);
        derivadaForma_xx(:,k)=jacTranspuesta\jacTranspuesta\derivadaForma_tt(:,k);
end

[matrizN]=matrizN60(N); % Matriz de 3x60    [ux,uy,uz]=matrizN*q(t)

derivadaForma_x=derivadaForma_x(1,:);      % Solo [...  dN_i/dx   ...]
derivadaForma_xx=derivadaForma_xx(1,:);    % Solo [...  d2N_i/dx2 ...]
```

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```
for j=1:20
    ind_q=1+3*(j-1); % Indice asociado a q nodales en direccion x (1,4,7, ...)
    matrizN_der_x(1,ind_q)=derivadaForma_x(j); % d^2uxr(xr,y,z)/dxr dtiempo = matrizN_der_x*qd
    matrizN_der_xx(1,ind_q)=derivadaForma_xx(j); % d^2 uxr(xr,y,z)/dxr^2 = matrizN_der_xx*q(t)
end
matrizN_der_x(3,60) =0; % 3x60 matrix
matrizN_der_xx(3,60)=0; % 3x60 matrix
dMelem=rho*matrizN'*matrizN*det(jacTranspuesta(:,:,1));              % dMelem
dGelem= -2*vo*rho*matrizN'*matrizN_der_x*det(jacTranspuesta(:,:,1)); % dGelem
dHelem=vo^2*rho*matrizN'*matrizN_der_xx*det(jacTranspuesta(:,:,1));  % dHelem

for i=1:60
    for j=1:60
        Me(i,j)=int(int(int(dMelem(i,j),t,-1,1),r,-1,1),s,-1,1);
        Ge(i,j)=int(int(int(dGelem(i,j),t,-1,1),r,-1,1),s,-1,1);
        He(i,j)=int(int(int(dHelem(i,j),t,-1,1),r,-1,1),s,-1,1);
    end
end
```

The symbolic expressions of matrices `Me`, `Ke`, `Ge`, and `He` were saved in individual functions with input parameters that can be called by a FE assembly function to assemble the global matrices M, K, G, and H of any 3-D mesh used for discretizing a finite portion of homogeneous soil. The code developed for moving 3-D finite-elements combined with beam FE and different vehicle models can be coupled to model a moving vehicle on rails using a finite domain.

Figure 6 shows the FE model of a moving load on a beam supported on a soil domain in relative coordinates analyzed with the developed program. A moving load at constant speed (shown in light green in the left figure) is assumed to be applied on the node with relative coordinates ($x = 0$, $y = 0$, and $z = 0$ in Figure 6). The FE model consists of 6177 nodes, 1200 20-node 3-D elastic FEs, and 60 2-node beam FEs. The beam is depicted in magenta and the elastic soil domain in blue. The load is assumed to be moving in the X-direction ($v_o = 15\, m/s$), the model parameters assumed for code-testing were `Soil.Poisson=0.30; Soil.G=200*1000000/9.81/1000; Soil.E=Soil.G*2*(1+Soil.Poisson); Soil.rho=2.6/9.81; Soil.gamma = 2.6 ton/m3, Soil.rho=gamma/g. Beam.rho=6*Soil.rho, Beam.Poisson=0.25; Beam.E=10000000; Beam.G=Beam.E/(2*(1+Beam.Poisson); Beam.A=0.1; Beam.Iz=0.1; Beam.Iy=0.1; Beam.J=Beam.Iz+Beam.Iy; Beam.Asy=5/6*Beam.A; Beam.Asz=5/6*Beam.A.`

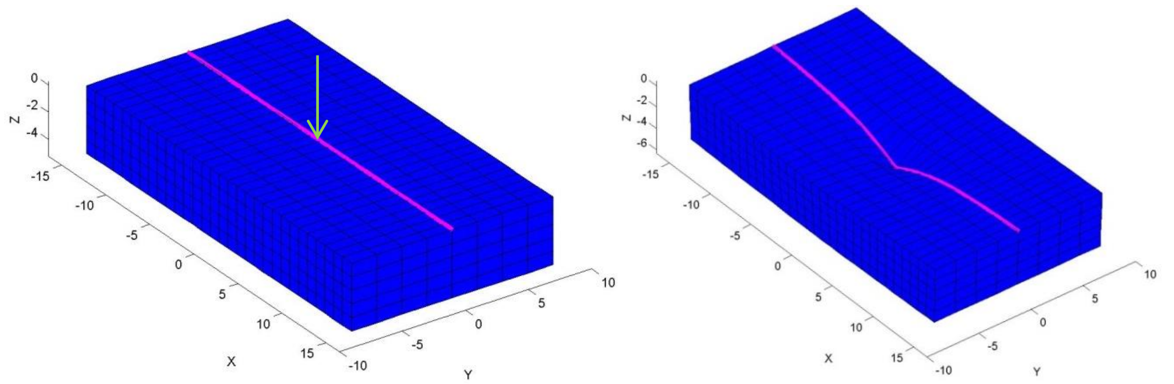

Figure 6. Left figure: Mesh of soil domain supporting beam with moving load/vehicle. Right figure: Stationary deformation for the analyzed case.

The nodes of the mesh have three displacements as degrees of freedom with the exception of the nodes of the beam model in contact with the soil mesh that have three displacements and three small rotations as degrees of freedom. The total number of dofs of the soil domain model assembled is 18531 and the total number of dofs is 18714 (including additional small rotations of beam nodes).

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The stationary deformation computed for the example case is show on the right (amplified to make it visible in Figure 6).

4 CONCLUSIONS AND FURTHER RESEARCH

The development of software tools for the estimation of response of moving vehicles, moving loads or moving masses on elastic homogeneous infinite domains has been presented. The use of moving meshes (formulation of displacement fields in relative coordinates) allows the construction of versatile models for the estimation of vehicle-induced vibrations with different applications requiring a significantly smaller mesh than that of conventional stationary finite elements. These tools can be applied for vibration-intensity estimation for environmental impact analysis of train or vehicle induced vibrations, including road or rail roughness using random vibration analysis. Automation in model generation for vehicles consisting in multiple cars (for train applications) moving at constant velocity on elastic rails will be approached in the near future. Other lines for future research are *i*) the feasibility of an homogenization strategy of periodic substructures such as sleepers under rails so that the proposed formulation can approximate the mechanical behavior moving vehicles on rails supported by sleepers and other periodic substructures using a moving mesh formulation, *ii*) a strategy for approximating the response of moving vehicles on non-homogeneous soil domains with stochastic elastic properties, and *iii*) the relevance of incorporating appropriate boundary layers in the moving mesh to allow absorbing boundaries.

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