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UNA FORMULACIÓN MIXTA PARA EL PROBLEMA DE POISSON FRACCIONARIO

A MIXED FORMULATION FOR THE FRACTIONAL POISSON PROBLEM

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Abstract. La formulación mixta del problema de Poisson clásico consiste en introducir un flujo como nueva variable con condiciones de borde adecuadas, obteniendo un sistema de ecuaciones acopladas. Usando identidades del cálculo fraccionario, en este trabajo exploramos una formulación mixta del problema de Poisson fraccionario y probamos que el problema está bien planteado. Una discretización directa del problema no parece posible, por lo que siguiendo ideas de Hughes y Masud introducimos una formulación estabilizada, que da lugar a un problema coercivo y bien planteado. La coercividad implica que cualquier discretización por elementos finitos conforme sea estable. Por último, obtenemos la convergencia de estas discretizaciones y discutimos su implementación.

Keywords: Fractional laplacian, mixed formulation, finite element method

Abstract. The mixed formulation of the classical Poisson problem consists in the introduction of a flux as a new variable with adequate boundary conditions, resulting in a system of coupled equation system. Using fractional calculus identities, in this work we explore a mixed formulation of the fractional Poisson problem and prove the well-posedness of the problem. A direct discretization of this problem seems out of reach, by following Hughes and Masud we are able to introduce a stabilized formulation that results in a coercive and well-posed problem. The coercivity implies that any confirming finite element discretization is stable. Lastly, we prove the convergence of this discretization and discuss its implementations.

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1 INTRODUCTION

In the last years, study of nonlocal operators has been an active area of research in different branches of mathematics. Nonlocal models have been increasingly used in different areas of science. Namely, machine learning (Rosasco et al. (2010), Lu et al. (2022), Wei et al. (2020)), finance (Carr et al. (2002)), image processing (Buades et al. (2010), Gilboa and Osher (2007), Lou et al. (2010)), magnetohydrodynamic (Schekochihin et al. (2008)), among others. In particular, the fractional Laplacian has been considered in many applications, including, for example, diffusion-reaction problems Yamamoto (2012), quasi-geostrophic flows Constantin and Wu (1999), transport in porous media De Pablo et al. (2012) and ultrasound Treeby and Cox (2010).

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. We propose to study a mixed formulation for the fractional Poisson problem in Ω, namely

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c := \mathbb{R}^d \setminus \Omega. \end{cases} \tag{1}$$

Above, $s \in (0,1)$, $f \in L^2(\Omega)$, and $(-\Delta)^s$ denotes the fractional Laplacian

$$(-\Delta)^s u(x) := C(d,s) \text{ p.v.} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy, \quad x \in \mathbb{R}^d, \tag{2}$$

where

$$C(d,s) := \frac{2^{2s} s \Gamma(s+\frac{d}{2})}{\pi^{d/2}\Gamma(1-s)}. \tag{3}$$

Note that $(-\Delta)^s$ is an operator of order $2s$. For the properties of this operator we refer to Acosta and Borthagaray (2017), Di Nezza et al. (2012), Lischke et al. (2019) and Daoud and Laamri (2022).

The fractional Laplacian can be regarded as a composition of certain *weighted,* nonlocal, vector calculus operators. Namely, given $w\colon \mathbb{R}^d \to \mathbb{R}$, we define its *fractional gradient of order* s, $\mathbf{grad}^s w\colon \mathbb{R}^d \to \mathbb{R}^d$,

$$\mathbf{grad}^s w(x) := \mu(d,s) \int_{\mathbb{R}^d} \frac{(w(x)-w(y))}{|x-y|^{d+s}} \frac{(x-y)}{|x-y|} dy, \tag{4}$$

and given $\mathbf{\Psi}\colon \mathbb{R}^d \to \mathbb{R}^d$ we define its *fractional divergence of order* s, $\mathrm{div}^s \mathbf{\Psi}\colon \mathbb{R}^d \to \mathbb{R}$,

$$\mathrm{div}^s \mathbf{\Psi}(x) := \mu(d,s) \int_{\mathbb{R}^d} \frac{(\mathbf{\Psi}(x)-\mathbf{\Psi}(y))}{|x-y|^{d+s}} \cdot \frac{(x-y)}{|x-y|} dy, \tag{5}$$

where

$$\mu(d,s) = \frac{2^s \Gamma(\frac{d+s+1}{2})}{\pi^{d/2}\Gamma(\frac{1-s}{2})}. \tag{6}$$

These operators possess the following properties. In first place, we have (e.g. D'Elia et al. (2021a))

$$(-\Delta)^s w = -\mathrm{div}^s \mathbf{grad}^s w. \tag{7}$$

Additionally, we have an integration by parts formula (e.g. Comi and Stefani (2019)): given $w \in C_c^\infty(\mathbb{R}^d)$, $\mathbf{\Psi} \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \mathbf{grad}^s w \cdot \mathbf{\Psi} = -\int_{\mathbb{R}^d} w \, \mathrm{div}^s \mathbf{\Psi}. \tag{8}$$

This formula can be extended to a broader class of functions (i.e. the spaces $\widetilde{H}^s(\Omega)$ and $\widetilde{L^2}(\Omega)$ defined below) via a density argument.

Some spaces we shall need are

$$\widetilde{H}^s(\Omega) := \{w \in H^s(\mathbb{R}^d) \colon \operatorname{supp} w \subset \overline{\Omega}\}, \tag{9}$$

furnished with the norm, cf. the Poincaré inequality (18),

$$\|w\|_{\widetilde{H}^s(\Omega)} := |w|_{H^s(\mathbb{R}^d)}, \tag{10}$$

where $H^s(\mathbb{R}^d)$ is the well know fractional Sobolev space

$$H^s(\mathbb{R}^d) := \{w \in L^2(\mathbb{R}^d) \colon |w|_{H^s(\mathbb{R}^d)} < \infty\} \tag{11}$$

and $|\cdot|_{H^s(\mathbb{R}^d)}$ is the Gagliardo semi-norm, cf. Di Nezza et al. (2012),

$$|w|_{H^s(\mathbb{R}^d)} := \left(\frac{C(d,s)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|w(x)-w(y)|^2}{|x-y|^{d+2s}}dx\,dy\right)^{\frac{1}{2}} = \|(-\Delta)^{\frac{s}{2}}w\|_{L^2(\mathbb{R}^d)}. \tag{12}$$

We also define

$$H(\operatorname{div}^s;\Omega) := \{\mathbf{\Psi} \in L^2(\mathbb{R}^d,\mathbb{R}^d) \colon (\operatorname{div}^s\mathbf{\Psi})\big|_\Omega \in L^2(\Omega)\}, \tag{13}$$

with the norm

$$\|\mathbf{\Psi}\|_{H(\operatorname{div}^s;\Omega)} := \left(\|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)} + \|(\operatorname{div}^s\mathbf{\Psi})\big|_\Omega\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{14}$$

We will denote by $\widetilde{L}^2(\Omega)$ the space of functions in $L^2(\Omega)$ that are extended by zero to Ω^c.

A crucial property for our analysis is the following lemma. It follows from the Parseval identity and an equivalent definition of $\mathbf{grad}^s$ for smooth functions via a convolution with the Riesz kernel

$$I_\alpha(x) = \frac{1}{c_\alpha}\frac{1}{|x|^{d-\alpha}}, \quad \alpha \in (0,d), \tag{15}$$

where

$$c_\alpha = \pi^{\frac{d}{2}}2^\alpha\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}. \tag{16}$$

This was shown in Shieh and Spector (2015).

Lemma 1 (Equivalence of seminorms)**.** *For a given $s \in (0,1)$, we have the equivalence of seminorms*

$$|w|_{H^s(\mathbb{R}^d)} \simeq \|\mathbf{grad}^s w\|_{L^2(\mathbb{R}^d)} \tag{17}$$

for all $w \in \widetilde{H}^s(\Omega)$.

Finally, we have the following Poincaré inequality, see (Edmunds and Evans, 2022, Theorem 3.9).

$$\|w\|_{L^2(\Omega)} \leq C_P|w|_{H^s(\mathbb{R}^d)} \simeq C_P\|\mathbf{grad}^s w\|_{L^2(\mathbb{R}^d)}, \quad \forall w \in \widetilde{H}^s(\Omega). \tag{18}$$

2 PROBLEM FORMULATION

The goal of this work is to consider a mixed formulation of the problem (1), which consists in the introduction of the flux as a new variable. Therefore, we consider the following fractional Darcy problem: find $(p, \mathbf{\Phi}) \in \widetilde{L}^2(\Omega) \times H(\mathrm{div}^s; \Omega)$ such that

$$\begin{cases} \mathbf{\Phi} + \mathbf{grad}^s p = 0 & \text{in } \mathbb{R}^d, \\ \mathrm{div}^s \mathbf{\Phi} = f & \text{in } \Omega, \\ p = 0 & \text{in } \Omega^c. \end{cases} \tag{19}$$

Clearly, $(p, \mathbf{\Phi})$ solves the problem above if and only if p solves (1) and $\mathbf{\Phi} = -\mathbf{grad}^s p$. We remark that, due to the nonlocal nature of the problem, this definition needs to be imposed in the whole space $\mathbb{R}^d$ and not just in the domain Ω.

Using the integration by parts formula (8), the weak formulation of (19) reads: find $(p, \mathbf{\Phi}) \in \widetilde{L}^2(\Omega) \times H(\mathrm{div}^s; \Omega)$ such that, for all $(q, \mathbf{\Psi}) \in \widetilde{L}^2(\Omega) \times H(\mathrm{div}^s; \Omega)$,

$$\int_{\mathbb{R}^d} \mathbf{\Phi} \cdot \mathbf{\Psi} - \int_{\mathbb{R}^d} p \, \mathrm{div}^s \mathbf{\Psi} + \int_{\mathbb{R}^d} q \, \mathrm{div}^s \mathbf{\Phi} = \int_{\mathbb{R}^d} f q. \tag{20}$$

We emphasize that all but the first of the integrals above need to be effectively computed in Ω.

The problem above has a clear saddle-point structure. We introduce some more notation. First, we define the forms

$$\begin{aligned} a \colon H(\mathrm{div}^s; \Omega) \times H(\mathrm{div}^s; \Omega) \to \mathbb{R}, &\quad a(\mathbf{\Phi}, \mathbf{\Psi}) = \int_{\mathbb{R}^d} \mathbf{\Phi} \cdot \mathbf{\Psi}, \\ b \colon \widetilde{L}^2(\Omega) \times H(\mathrm{div}^s; \Omega) \to \mathbb{R}, &\quad b(q, \mathbf{\Psi}) = \int_{\Omega} q \, \mathrm{div}^s \mathbf{\Psi}, \\ F \colon \widetilde{L}^2(\Omega) \to \mathbb{R}, &\quad F(q) = \int_{\Omega} f q. \end{aligned} \tag{21}$$

Finally, we introduce

$$B \colon H(\mathrm{div}^s; \Omega) \to \widetilde{L}^2(\Omega) \tag{22}$$

by its Riesz representative,

$$(B\mathbf{\Psi}, q)_{L^2(\Omega)} := b(q, \mathbf{\Psi}), \quad \forall \mathbf{\Psi} \in H(\mathrm{div}^s; \Omega), \ q \in \widetilde{L}^2(\Omega). \tag{23}$$

3 WELL-POSEDNESS

Notice that a is symmetric. By standard arguments in the analysis of mixed formulations, to prove the well-posedness of (20) it suffices to show that

- the form a is coercive in $\ker B$;
- the form b satisfies an inf-sup condition.

The fact that a is coercive in $\ker B$ follows straightforwardly upon observing that $\ker B = \{\mathbf{\Psi} \in H(\mathrm{div}^s; \Omega) \colon \mathrm{div}^s \mathbf{\Psi} = 0 \text{ in } \Omega\}$. This yields that, for every $\mathbf{\Psi} \in \ker B$,

$$a(\mathbf{\Psi}, \mathbf{\Psi}) = \|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)} = \|\mathbf{\Psi}\|^2_{H(\mathrm{div}^s; \Omega)}. \tag{24}$$

The inf-sup condition for b follows by the surjectivity of the div^s operator.

Lemma 2 (inf-sup condition)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. The map $\operatorname{div}^s|_\Omega$ such that $\operatorname{div}^s|_\Omega \mathbf{\Psi} := (\operatorname{div}^s \mathbf{\Psi})|_\Omega$ maps $H(\operatorname{div}^s;\Omega)$ onto $L^2(\Omega)$. Consequently, b satisfies an inf-sup condition: there exists $\beta > 0$ such that*

$$\inf_{p\in\widetilde{L}^2(\Omega)} \sup_{\mathbf{\Phi}\in H(\operatorname{div}^s;\Omega)} \frac{b(p,\mathbf{\Phi})}{\|p\|_{L^2(\Omega)}\|\mathbf{\Phi}\|_{H(\operatorname{div}^s;\Omega)}} \geq \beta. \tag{25}$$

As a corollary, we deduce the well-posedness of the fractional Darcy problem (Boffi et al., 2013, Theorem 4.2.3).

Proposition 1 (well-posedness)**.** *Problem (20) has a unique solution $(p,\mathbf{\Phi}) \in \widetilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)$, and there hold*

$$\begin{aligned} \|p\|_{L^2(\Omega)} &\leq (1+C_P^2)\|f\|_{L^2(\Omega)}, \\ \|\mathbf{\Phi}\|_{H(\operatorname{div}^s;\Omega)} &\leq 2\sqrt{1+C_P^2}\|f\|_{L^2(\Omega)}. \end{aligned} \tag{26}$$

4 STABILIZATION

Here, we address finite element approximations of (20). A direct discretization of such a problem is out of reach, because the construction of $H(\operatorname{div}^s)$-conforming finite elements a-la Raviart-Thomas seems unfeasible. Instead, here we follow Masud and Hughes (2002) and pursue the use of a stabilized method.

To shorten the notation, we write

$$\begin{aligned} &\mathcal{L}\colon \left(\widetilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)\right)\times\left(\widetilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)\right)\to\mathbb{R}, \\ &\mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) := a(\mathbf{\Phi},\mathbf{\Psi}) - b(p,\mathbf{\Psi}) + b(q,\mathbf{\Phi}), \end{aligned} \tag{27}$$

so that we can rewrite (20) as: find $(p,\mathbf{\Phi}) \in \widetilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)$ such that

$$\mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) = F(q), \quad \forall (q,\mathbf{\Psi}) \in \widetilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega). \tag{28}$$

We introduce the stabilized form in $\mathbb{V} := \widetilde{H}^s(\Omega)\times H(\operatorname{div}^s;\Omega)$,

$$\begin{aligned} &\mathcal{L}_{\text{stab}}\colon \mathbb{V}\times\mathbb{V}\to\mathbb{R}, \\ &\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) := \mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) + \frac{1}{2}\int_{\mathbb{R}^d} (\mathbf{\Phi}+\mathbf{grad}^s p)\cdot(-\mathbf{\Psi}+\mathbf{grad}^s q)\,. \end{aligned} \tag{29}$$

With this, we consider the stabilized problem: find $(p,\mathbf{\Phi}) \in \mathbb{V}$ such that

$$\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) = F(q) \quad \forall (q,\mathbf{\Psi}) \in \mathbb{V}. \tag{30}$$

We make two important remarks regarding the definition of $\mathcal{L}_{\text{stab}}$. First, we have shrunk the domain by replacing $\widetilde{L}^2(\Omega)$ by $\widetilde{H}^s(\Omega)$ so that the stabilization term is well-defined. Second, the stabilization term above involves integration on the whole $\mathbb{R}^d$. Let us consider the following norm in $\mathbb{V}$,

$$|||(q,\mathbf{\Psi})||| := \left[\frac{1}{2}\left(\|\mathbf{grad}^s q\|^2_{L^2(\mathbb{R}^d)} + \|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)}\right)\right]^{1/2}. \tag{31}$$

Remark 1 (equivalence)**.** *A pair $(p,\mathbf{\Phi}) \in \mathbb{V}$ solves the problem (30) if and only if it solves (28).*

Lemma 3 (stability/coercivity)**.** *We have*

$$\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(p,\mathbf{\Phi})) = |||(p,\mathbf{\Phi})|||^2 \quad \forall (p,\mathbf{\Phi}) \in \mathbb{V}. \tag{32}$$

Proof. Indeed, if $(p,\mathbf{\Phi}) \in \mathbb{V}$ then

$$\begin{aligned}\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(p,\mathbf{\Phi})) &= \|\mathbf{\Phi}\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\int_{\mathbb{R}^d} (\mathbf{\Phi} + \mathbf{grad}^s p)\cdot(-\mathbf{\Phi} + \mathbf{grad}^s p) \\ &= \frac{1}{2}\|\mathbf{\Phi}\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\mathbf{grad}^s p\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

□

Lemma 4 (continuity)**.** *We have*

$$\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) \le |||(p,\mathbf{\Phi})|||\,|||(q,\mathbf{\Psi})||| \quad \forall (p,\mathbf{\Phi}),(q,\mathbf{\Psi}) \in \mathbb{V}. \tag{33}$$

Proof. Let $(p,\mathbf{\Phi}),(q,\mathbf{\Psi}) \in \mathbb{V}$. Using the integration by parts formula (8) we can rewrite the stabilized form as

$$\begin{aligned}|\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi}))| =& \Bigg|\frac{1}{2}\int_{\mathbb{R}^d} \mathbf{\Phi}\cdot\mathbf{\Psi} + \frac{1}{2}\int_{\mathbb{R}^d} \mathbf{grad}^s p\cdot\mathbf{\Psi} - \frac{1}{2}\int_{\mathbb{R}^d} \mathbf{grad}^s q\cdot\mathbf{\Phi} \\ &+ \frac{1}{2}\int_{\mathbb{R}^d} \mathbf{grad}^s p\cdot\mathbf{grad}^s q\Bigg|,\end{aligned} \tag{34}$$

therefore by the inequality $(a+b)^2 \le 2(a^2+b^2)$ we deduce

$$\begin{aligned}|\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi}))| &\le \frac{1}{2}\left(\|\mathbf{\Phi}\|_{L^2(\mathbb{R}^d)} + \|\mathbf{grad}^s p\|_{L^2(\mathbb{R}^d)}\right)\left(\|\mathbf{\Psi}\|_{L^2(\mathbb{R}^d)} + \|\mathbf{grad}^s q\|_{L^2(\mathbb{R}^d)}\right) \\ &\le |||(p,\mathbf{\Phi})|||\,|||(q,\mathbf{\Psi})|||.\end{aligned} \tag{35}$$

□

As usual, the two lemmas above and the Lax-Milgram theorem give rise to the well posedness of our problem.

Proposition 2 (well-posedness of stabilized formulation)**.** *Given $f \in L^2(\Omega)$, problem (30) has a unique solution $(p,\mathbf{\Phi}) \in \mathbb{V}$. Moreover, we have the stability estimate*

$$|||(p,\mathbf{\Phi}))||| \le C_P\sqrt{2}\|f\|_{L^2(\Omega)}. \tag{36}$$

Sobolev regularity up to $\partial\Omega$ of u, the solution of (1), was established in Borthagaray and Nochetto (2023); for f in the Besov space $B^{-s+1/2}_{2,1}(\Omega)$, the solution u lies in the space $\cap_{\varepsilon>0}\widetilde{H}^{s+1/2-\varepsilon}(\Omega)$. Note that the hypothesis on f is weaker than L^2 when $s > 1/2$; if $f \in L^2(\Omega)$ and $s \le 1/2$, we have $u \in \cap_{\varepsilon>0}\widetilde{H}^{2s-\varepsilon}(\Omega)$.

Moreover, we deduce a regularity estimate for the flux by means of the mapping properties of $\mathbf{grad}^s$; if $u \in \widetilde{H}^{r+s}(\Omega)$ then $\mathbf{grad}^s u \in H^r(\mathbb{R}^d;\mathbb{R}^d) := (H^r(\mathbb{R}^d))^d$, cf. D'Elia et al. (2021b). We summarize this discussion in the following proposition.

Proposition 3 (Regularity of the solutions)**.** *Given* $f \in L^2(\Omega)$. *Consider* $(p, \mathbf{\Phi}) \in \mathbb{V}$ *the solution of* (30). *We have*

$$\begin{aligned}
\|p\|_{\widetilde{H}^{s+\frac{1}{2}-\varepsilon}(\Omega)} + |\mathbf{\Phi}|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^d;\mathbb{R}^d)} &\leq \frac{C}{\sqrt{\varepsilon|1-2s|}}\|f\|_{L^2(\Omega)}, && \textit{for } s > \frac{1}{2}, \\
\|p\|_{\widetilde{H}^{2s-\varepsilon}(\Omega)} + |\mathbf{\Phi}|_{H^{s-\varepsilon}(\mathbb{R}^d;\mathbb{R}^d)} &\leq \frac{C}{\sqrt{\varepsilon|1-2s|}}\|f\|_{L^2(\Omega)}, && \textit{for } s < \frac{1}{2}, \\
\|p\|_{\widetilde{H}^{1-\varepsilon}(\Omega)} + |\mathbf{\Phi}|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^d;\mathbb{R}^d)} &\leq \frac{C}{\varepsilon}\|f\|_{L^2(\Omega)}, && \textit{for } s = \frac{1}{2},
\end{aligned} \tag{37}$$

for any $\varepsilon < \max\{\min\{2s, \frac{1}{2}+s\}, \frac{1}{4}\}$.

5 FINITE ELEMENT DISCRETIZATION

By replacing $\mathcal{L}$ with $\mathcal{L}_{\text{stab}}$ we have obtained a coercive formulation. Therefore, if we take any conforming finite element space, we immediately obtain a stable discretization. For the sake of this work, we shall consider continuous, piecewise linear discretizations.

Let us then begin by describing the discrete framework that we will use, we closely follow section 4 of Borthagaray et al. (2019). Notice that we are approximating $\mathbf{\Phi}$, which is not compactly supported, and the form a in (21) and the stabilization term in (29) involve integration in $\mathbb{R}^d$. In order to tackle this problem we will consider a ball B_H containing Ω and such that $H := d(\overline{\Omega}, B_H^c)$.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of simplicial triangulations of $\overline{B_H}$ which we will assume regular, i.e., there exists a constant $c > 0$ such that

$$\sup_{h>0} \sup_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T} = c, \tag{38}$$

where $h_T = diam(T)$ and ρ_T is the diameter of the largest ball contained in T. We also assume that the set $\{T \in \mathcal{T}_h : T \cap \Omega \neq \emptyset\}$ is a simplicial triangulation of $\overline{\Omega}$. The nodes of $\mathcal{T}_h$ will be denoted by $\{z_i\}$. On the triangulation $\mathcal{T}_h$ we define $\mathbb{V}_h$ as the functions $(q_h, \mathbf{\Psi}_h) \in \mathcal{P}_1(\mathcal{T}_h) \times \mathcal{P}_1^d(\mathcal{T}_h) \subset \mathbb{V}$ such that q_h vanishes on Ω^c and $\mathbf{\Psi}_h$ vanishes on B_H^c. As usual, we introduce the Lagrange nodal basis $\{\varphi_i\}$, corresponding to the internal nodes $\{z_i\}$. We denote by B_i the largest ball centered in z_i and contained in $\text{supp}(\varphi_i)$. We consider the following discrete problem: find $(p_h, \mathbf{\Phi}_h) \in \mathbb{V}_h$ such that

$$\mathcal{L}_{\text{stab}}((p_h, \mathbf{\Phi}_h), (q_h, \mathbf{\Psi}_h)) = F(q_h) \quad \forall (q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h. \tag{39}$$

The coercive formulation and the fact that $\mathbb{V}_h \subset \mathbb{V}$ imply the existence and uniqueness of solutions to (39) and the Galerkin orthogonality.

Proposition 4 (best approximation)**.** *Let* $(p, \mathbf{\Phi}) \in \mathbb{V}$ *and* $(p_h, \mathbf{\Phi}_h) \in \mathbb{V}_h$ *be the solutions to* (30) *and* (39)*, respectively. We have the following Galerkin orthogonality:*

$$\mathcal{L}_{\text{stab}}((p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h), (q_h, \mathbf{\Psi}_h)) = 0 \quad \forall (q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h. \tag{40}$$

Consequently, we obtain

$$|||(p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)||| \leq \inf_{(q_h, \mathbf{\Psi}_h)\in\mathbb{V}_h} |||(p - q_h, \mathbf{\Phi} - \mathbf{\Psi}_h)|||. \tag{41}$$

To get convergence estimates, we will need interpolation estimates. The usual Lagrange interpolation does not seem a feasible option in our setting because of the low regularity of our solutions and the lack of stability and approximation properties with respect to the corresponding low-order fractional Sobolev spaces. Therefore, we will use the quasi-interpolation operators Π_h and $\mathbf{\Pi}_h$ introduced in Chen and Nochetto (2000).

Definition 1 (quasi-interpolation operators)**.** *We define* $\Pi_h : \widetilde{H}^s(\Omega) \to \mathcal{P}_1(\mathcal{T}_h)$ *and* $\mathbf{\Pi}_h : H(\mathrm{div}^s;\Omega) \to \mathcal{P}_1^d(\mathcal{T}_h)$ *as*

$$\begin{aligned}\Pi_h q &:= \sum_{z_i \in \Omega} \left(\frac{1}{|B_i|} \int_{B_i} q(x)\, dx \right) \varphi_i, \\ \mathbf{\Pi}_h \mathbf{\Psi} &:= \sum_{z_i \in B_H} \left(\frac{1}{|B_i|} \int_{B_i} \mathbf{\Psi}(x)\, dx \right) \varphi_i. \end{aligned} \tag{42}$$

We refer to Chen and Nochetto (2000) and Borthagaray et al. (2019) for the properties of this operator. For a regular family of triangulations, we have the following global interpolation estimates,

$$\begin{aligned}\|q - \Pi_h q\|_{\widetilde{H}^s(\Omega)} &\le C(d,c,t) h^{t-s} \|q\|_{\widetilde{H}^t(\Omega)}, \\ \|\mathbf{\Psi} - \mathbf{\Pi}_h \mathbf{\Psi}\|_{L^2(B_H;\mathbb{R}^d)} &\le C(d,c,t) h^t |\mathbf{\Psi}|_{H^t(\mathbb{R}^d;\mathbb{R}^d)}, \end{aligned} \tag{43}$$

for $0 < t < 2$. The following lemma gives us a global interpolation estimate for the flux in $\mathbb{R}^d$. For the sake of simplicity we assume that the origin is contained in Ω and that B_H is centered in the origin.

Lemma 5 (Global interpolation estimates for the flux)**.** *Let* $(p, \mathbf{\Phi}) \in \mathbb{V}$ *be the solution to* (30) *and* $t \in (0, 2)$. *We have,*

$$\begin{aligned}\|\mathbf{\Phi} - \mathbf{\Pi}_h \mathbf{\Phi}\|_{L^2(B_H;\mathbb{R}^d)} &\le C(d,\Omega,t) h^t |\mathbf{\Psi}|_{H^t(\mathbb{R}^d;\mathbb{R}^d)}, \\ \|\mathbf{\Phi} - \mathbf{\Pi}_h \mathbf{\Phi}\|_{L^2(B_H^c;\mathbb{R}^d)} &\le C(d,s,\Omega) H^{-\frac{1}{2}-\frac{d}{2}-s} \|p\|_{L^2(\Omega)}. \end{aligned} \tag{44}$$

Proof. The first inequality is (43).

For the complement of the ball, as $\mathbf{\Pi}_h \mathbf{\Phi}|_{B_H^c} \equiv 0$ and $\mathbf{grad}^s p = \mathbf{\Phi}$, we have

$$\|\mathbf{\Phi} - \mathbf{\Pi}_h \mathbf{\Phi}\|_{L^2(B_H^c;\mathbb{R}^d)} = \|\mathbf{\Phi}\|_{L^2(B_H^c;\mathbb{R}^d)} = \|\mathbf{grad}^s p\|_{L^2(B_H^c;\mathbb{R}^d)}.$$

Given $x \in B_H^c$, by the definition of $\mathbf{grad}^s$ and the Poincaré inequality (18), we deduce

$$\begin{aligned}|\mathbf{grad}^s p(x)|^2 &= \left| \mu(d,s) \int_\Omega p(y) \frac{x-y}{|x-y|^{d+s+1}} dy \right|^2 \\ &\le C_P^2 \mu(d,s)^2 |\Omega| \|p\|_{L^2(\Omega)}^2 \frac{1}{d(x,\Omega)^{2(d+s)}}. \end{aligned} \tag{45}$$

Therefore, by the inequality $d(x,\Omega) \ge |x| - diam(\Omega)$ (recall that we are assuming $0 \in \Omega$) and a change of variables to polar coordinates, we deduce the bound

$$\begin{aligned}\|\mathbf{grad}^s p\|_{L^2(B_H^c;\mathbb{R}^d)}^2 &\le C\mu(d,s)^2 \, \|p\|_{L^2(\Omega)}^2 \int_{B_H^c} \frac{1}{d(x,\Omega)^{2(d+s)}} dx \\ &\le C\mu(d,s)^2 H^{-1-d-2s} |\Omega| \, \|p\|_{L^2(\Omega)}^2. \end{aligned} \tag{46}$$

Here we are using the fact that for sufficiently large ball B_H, its radius and H are comparable. $\square$

As a corollary we obtain convergence rates for the numerical scheme.

Corollary 1 (Order of convergence)**.** *Let* $(p, \mathbf{\Phi}) \in \mathbb{V}$ *and* $(p_h, \mathbf{\Phi}_h) \in \mathbb{V}_h$ *be the solutions to* (30) *and* (39)*, respectively. We have the following order of convergence,*

$$|||(p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)||| \leq \begin{cases} Ch^{\frac{1}{2}}|\log h|^{\frac{1}{2}}\|f\|_{L^2(\Omega)}, & \textit{for } s > \frac{1}{2}, \\ Ch^{s}|\log h|^{\frac{1}{2}}\|f\|_{L^2(\Omega)}, & \textit{for } s < \frac{1}{2}, \\ Ch^{\frac{1}{2}}|\log h|\,\|f\|_{L^2(\Omega)}, & \textit{for } s = \frac{1}{2}. \end{cases} \tag{47}$$

Proof. We assume $s > \frac{1}{2}$, the other cases follow by the same argument.

By combining the best approximation property (cf. Proposition 4) with the interpolation estimates (43) and the Lemma 5, we obtain

$$|||(p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)|||^2 \leq C\left(h^{2t-2s}\|p\|^2_{\widetilde{H}^t(\Omega)} + h^{2t}|\mathbf{\Phi}|^2_{H^t(\mathbb{R}^d;\mathbb{R}^d)} + H^{-1-d-2s}\|p\|^2_{L^2(\Omega)}\right), \tag{48}$$

for $t \in (0, 2)$. Now, by choosing $t = s + \frac{1}{2} - \varepsilon$ and $\varepsilon = |\log h|^{-1}$ (such that $h^{-\varepsilon}$ is constant for $h < 1$), we can estimate the first term in the right hand side of (48) by means of the regularity estimates (cf. Proposition 3);

$$h^{2t-2s}\|p\|^2_{\widetilde{H}^t(\Omega)} \leq Ch|\log h|\|f\|^2_{L^2(\Omega)}. \tag{49}$$

The same argument with $t = \frac{1}{2} - \varepsilon$ and $\varepsilon = |\log h|^{-1}$ shows

$$h^{2t}|\mathbf{\Phi}|^2_{H^t(\mathbb{R}^d;\mathbb{R}^d)} \leq Ch|\log h|\|f\|^2_{L^2(\Omega)}. \tag{50}$$

For the third term, the choice of $H^{-1-d-2s} \simeq h|\log h|$ and the stability properties (cf. Proposition 2) yields

$$H^{-1-d-2s}\|p\|^2_{L^2(\Omega)} \leq Ch|\log h|\|f\|^2_{L^2(\Omega)}. \tag{51}$$

$\square$

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